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THE UNIVERSITY OF ALBERTA

ON THE NON-LINEAR ELECTROMAGNETIC THEORY

$$F_{ik,k} = - (1 + A_k^2) A_i$$

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, the thesis entitled ON THE NON-LINEAR ELECTROMAGNETIC THEORY $F_{ik,k} = -(1 + A_k^2) A_i$, submitted by George W. Darewych in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

The non-linear electromagnetic theory proposed by Dr. Schiff is investigated. Non-Maxwellian plane wave solutions of the field equations, $F_{ik,k} = - (1 + A_k^2) A_i$, are obtained. Static, particlelike solutions of the field equations are considered in the absence of a magnetic field. Existence of well behaved solutions, in the case of spherical symmetry, is formally established. The list of known spherically symmetric, neutral and charged particlelike solutions is extended to include "compound" solutions which correspond to an imaginary vector potential in certain (linear) regions of space. Sufficient conditions are derived under which variational approximations yield upper bounds to the Lagrangian associated with the class of field equations $\Delta\phi = F'(\phi)$. The usual parameter variation method is generalised, in the case of more than one dimension, to admit variation of functions of one of the independent variables. This method is used to obtain approximations to possible nonspherical, odd parity, neutral particlelike eigensolutions of $\Delta\phi = \phi - \phi^3$. The results, however, seem to give an approximation to the lowest two-particle state rather than a one-particle state of odd parity. A variational principle for charged particlelike solutions is developed and used to obtain approximations to the lowest (in energy) spherically symmetric state. A scheme is suggested whereby integral relations, which are satisfied by integrable solutions of $\Delta\phi = F'(\phi)$, may be generated. The first few such integral relations are derived

and are used, in the particular case $F'(\phi) = \phi - \phi^3$, to obtain alternate expressions for the energy, $\int[\frac{1}{2}(\nabla\phi)^2+F(\phi)]d^3x$, of the system and to test variational approximations to the solutions of the field equation. The dynamical stability of neutral particlelike solutions, against certain dissociation modes, is demonstrated; in particular, against direct dissociation into plane waves and into particles for which $\vec{H} = 0$.

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I. INTRODUCTION

One of the basic problems which arises before every generation of physicists is that of "elementary particles." The physicist attempts to determine what constitutes the basic "building blocks" of matter, how they arise and what are their properties.

With the discovery in recent years of a great number of what are presently considered to be "elementary particles"[1] and their resonances, this problem is very much in the forefront of research both experimentally and theoretically. On the theoretical side, quantum field theory provides a good tool for describing the nature of the interactions between particles.[2]

Recently, the investigation of symmetries among the various properties of the particles (charge, spin, parity, isotopic spin, etc.) has led to an apparently successful method of classifying these particles in symmetry groups.[3]

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1. A reasonably up-to-date list of observed particles and their properties is given by A.M. Baldin and A.A. Komar in Sov. Phys.: Uspekhi, 8, p. 307, Sep. 1965.

2. There is a large number of good texts on this subject, among them references (1), (2), (3) of Bibliography.

3. References (4), (5), and (6), give a good account of the present state of this rapidly changing subject.

This in turn has led to the prediction of new, hitherto unobserved particles such as the Ω^-, as well as new information about the properties of "old" ones. Both these approaches, however, do not lend themselves to explaining the intrinsic nature of elementary particles, how they arise and why with the particular values of such basic parameters characterising them (such as mass and charge) as are observed in nature. One approach which hopefully can shed some light on this problem forms the subject of this thesis. We consider here the classical non-linear electromagnetic theory proposed by Dr. Schiff (7). In this theory the basic entity is the electromagnetic field, while particles (spinless bosons) are manifestations of this field arising through the mechanism of a self interaction term in the Lagrangian and hence the field equations.

This approach dates back to the attempts of Mie[1] to modify Maxwell's electromagnetic field equations in a way to account for the existence of particles. The hope was that the modified field equations would yield solutions which differ from those of Maxwell's equations principally in a localised region of space corresponding to the "interior" of the particle. One of the most widely investigated examples of Mie's theory, based on the Lagrangian

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1. G. Mie, Ann. Physik <u>37</u>, p. 511 (1912); <u>39</u>, p. 1 (1912); <u>40</u> p. 1, (1913). A good account of this theory and other attempts is given by Teshima (8).

$$\mathcal{L} = b^2 \left[\sqrt{1 - \frac{1}{8\pi b^2} F_{ij}^{\ 2}} - 1 \right],$$

was considered by Born and Infeld.[1]

The immediate object of investigation in such theories is the determination of particle-like solutions of the field equations --- solutions for which the parameters of the field and all physically observable quantities derivable from them are well behaved.[2] The purpose of this thesis is to examine further the theory proposed by Dr. Schiff (7). An appreciable amount of work has already been done on this subject[3], and the more important aspects of it will be abstracted throughout this thesis as required for completeness.

The proposed field equations are (in the usual four-notation):

$$F_{ik,k} = - \left[\left(\frac{mc}{\hbar}\right)^2 + \left(\frac{g}{\hbar c}\right)^2 A_k^{\ 2} \right] A_i \qquad \text{(I-1)}$$

where $F_{ik} = A_{k,i} - A_{i,k}$,

$A_k \equiv (\vec{A}, i\phi)$,

and $x_k \equiv (\vec{x}, ict)$.

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1. M. Born, Proc. Roy. Soc. (London), A143, p. 410, (1934); M. Born and L. Infeld, ibid., 144, p. 425 (1934), 147, p. 522, (1934); 150, p. 41 (1935).
2. Throughout this thesis, "good behaviour" will imply existence, continuity and single valuedness throughout the region under consideration.
3. References (7), (8), (9), (10).

$\vec{A}$ and Φ are the usual vector and scalar potentials respectively, while m and g are universal constants having dimensions of mass and charge respectively. The summation convention on repeated indices will be used throughout, with $i, j, k, \ldots = 1, 2, 3, 4$ while $\alpha, \beta, \gamma, \ldots = 1, 2, 3$. Also $f_{;k} \equiv \frac{\partial f}{\partial x_k}$.

The right hand side of (I-1) is identified as the electric 4-current, which, because of the antisymmetry of F_{ik}, is clearly conserved.

Under the transformation $A_i \rightarrow A_i \cdot \frac{mc^2}{g}$ and $x_i \rightarrow x_i \cdot \frac{\hbar}{mc}$, the field equations take on the convenient (dimensionless) form:

$$F_{ik,k} = -(1 + A_k^{\,2})\, A_i \quad . \tag{I-1a}$$

In vector notation, since

$$F_{ik} = \begin{bmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix} ,$$

the field equations are:

$$\nabla x\vec{H} = \frac{\partial\vec{E}}{\partial t} - (1 + \vec{A}^2 - \phi^2)\,\vec{A}$$

$$\nabla\cdot\vec{E} = -(1 + \vec{A}^2 - \phi^2)\phi \qquad \text{(I-1b)}$$

The theory (as is obvious from (I-1)) is relativistically covariant, but is not gauge invariant. The conservation of the 4-current imposes a gauge condition on the 4-potential

$$\frac{\partial}{\partial x_i}\,[\,(1 + A_k{}^2)\,A_i\,] = 0 \quad ,$$

which will be referred to as the "generalised Lorentz gauge." In vector notation:

$$(\vec{A}\cdot\nabla + \phi\frac{\partial}{\partial t})(1 + \vec{A}^2 - \phi^2) + (1 + \vec{A}^2 - \phi^2)(\nabla\cdot\vec{A} + \frac{\partial\phi}{\partial t}) = 0.$$

Any solution of the field equations (I-1a) will necessarily satisfy this gauge condition. We note, however, that under the gauge $A_k{}^2 = -1$ one returns to the Maxwell free field equations. Except for this case, the field equations are non-linear, so the principle of superposition is not generally applicable. As a corollary this implies that any (stable) particle will, in general, decay in a constant potential field.

The field equations (I-1) are equivalent to the principle of least action

$$\delta S = 0 ,$$

with the action function defined as

$$S = \int \mathcal{L}(x_i, A_i, A_{i,k})\, d^4x \quad ,$$ [1]

where the Lagrangian density is

$$\mathcal{L} = -\frac{N}{4\pi}\left[\frac{1}{4}F_{ik}^{\;2} + \frac{1}{2}A_i^{\;2} + \frac{1}{4}A_i^{\;2}A_k^{\;2} + K\right] \qquad \text{(I-2)}$$

K and N being constants.

The symmetrised energy-momentum tensor is

$$T_{ik} = -\frac{N}{4\pi}\Big[\left(\frac{1}{4}F_{\ell m}^{\;2} + \frac{1}{2}A_\ell^{\;2} + \frac{1}{4}A_\ell^{\;2}A_m^{\;2} + K\right)\delta_{ik}$$

$$- F_{i\ell}F_{k\ell} - (1 + A_\ell^{\;2})A_iA_k\Big] \quad . \qquad \text{(I-3)}$$ [2]

It satisfies the usual conservation law

$$T_{ik,k} = 0 \quad . \qquad \text{(I-4)}$$

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1 Integration over the entire range of values of the independent variables x_i is implied throughout, unless otherwise specified. $d^4x \equiv dx_1\,dx_2\,dx_3\,dx_4$ throughout.

2 Components of the symmetrised energy-momentum tensor are listed in appendix A. (I-3) is obtainable, to within a divergenceless term of the form $(A_iF_{k\ell})_{,\ell}$, from the usual definition $\mathcal{L}\delta_{ik} - A_{\ell,i}\dfrac{\partial\mathcal{L}}{\partial A_{\ell,k}}$.

The physical implication of these (four) equations is that the four-momentum

$$P_i = -i \int_{\Sigma} T_{ik} \, ds_k \quad ,$$

where Σ is any spacelike hypersurface of infinite extension, is conserved. $P_i \equiv (\vec{P}, iW)$ and $\vec{P}$ is the total (relativistic) momentum and W the energy of the system.

Since the largest transformation group under which the field equations (I-1) are invariant is the ten parameter inhomogeneous Lorentz group[1], then according to Noether's theorem[2] there should be a total of ten conservation laws. The four conservation laws (I-4) are a consequence of invariance under four-translations. Invariance under four rotations (the homogeneous Lorentz group) leads to the six conservation laws

$$M_{k\ell m,k} = 0 \quad , \qquad \text{(I-5)}$$

where

$$M_{k\ell m} = (T'_{k\ell} x_m - T'_{km} x_\ell) + \left(\frac{\partial \mathcal{L}}{\partial A_{\ell,k}} A_m - \frac{\partial \mathcal{L}}{\partial A_{m,k}} A_\ell\right)$$

$$= L_{k\ell m} + S_{k\ell m} \qquad \text{(I-6)}$$

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1 The group of translations and rotations in Minkowski four-space. For a definition see (11) p. 32.

2 (11), p. 54.

is the "total angular momentum" density. $T'_{k\ell}$ is the "canonical energy-momentum tensor" :

$$T'_{ik} = T_{ik} + \frac{N}{4\pi} (F_{i\ell} A_k)_{,\ell} \;. \qquad \text{(I-7)}$$

The tensors
$$L_{k\ell m} = T'_{k\ell} x_m - T'_{km} x_\ell \;, \qquad \text{(I-8)}$$

$$S_{k\ell m} = \frac{\partial \mathcal{L}}{\partial A_{\ell,k}} A_m - \frac{\partial \mathcal{L}}{\partial A_{m,k}} A_\ell$$
$$= - \frac{N}{4\pi} (F_{k\ell} A_m - F_{km} A_\ell) \;,$$

may be interpreted as the "orbital angular momentum" density and "intrinsic" or "spin angular momentum" density respectively. $L_{k\ell m}$ and $S_{k\ell m}$ do not separately satisfy equations of the form (I-5), hence the orbital angular momentum

$$L_{\ell m} = -i \int_\Sigma L_{k\ell m} \, ds_k$$

and the spin

$$S_{\ell m} = -i \int_\Sigma S_{k\ell m} \, ds_k$$

are not conserved quantities, that is their values depend on the choice of Σ. Nevertheless, if Σ is taken to be the hypersurface X_4 = const., then the spin is

$$S_{\ell m} = -i \int S_{4\ell m} \, d^3x$$

where

$$S_{4\ell m} = \frac{N}{4\pi}\begin{bmatrix} 0 & i(\vec{A}x\vec{E})_z & -i(\vec{A}x\vec{E})_y & E_x\phi \\ -i(\vec{A}x\vec{E})_z & 0 & i(\vec{A}xE)_x & E_y\phi \\ i(\vec{A}x\vec{E})_y & -i(\vec{A}x\vec{E})_x & 0 & E_z\phi \\ -E_x\phi & -E_y\phi & -E_z\phi & 0 \end{bmatrix} \qquad \text{(I-9)}$$

This is identical in form to the corresponding expression for a free Maxwell field (11).

The orbital angular momentum may be re-defined to satisfy a conservation law by using the symmetrised in place of the canonical energy-momentum tensor in (I-8):

$$J_{k\ell m} = T_{k\ell}x_m - T_{km}x_\ell \quad .$$

Since $T_{k\ell} = T_{\ell k}$, we have $J_{k\ell m,k} = 0$. Such a definition obscures the intrinsic or "spin" part of the total angular momentum density, since from (I-8) and (I-7) we obtain

$$M_{k\ell m} = (T_{k\ell}x_m - T_{km}x_\ell) + \frac{N}{4\pi}\left[(F_{ki}A_\ell)_{,i}x_m - (F_{ki}A_m)_{,i}x_\ell\right] - \frac{N}{4\pi}(F_{k\ell}A_m - F_{km}A_\ell)$$

$$= J_{k\ell m} + \frac{N}{4\pi}\left[F_{ki}(A_\ell x_m - A_m x_\ell)\right]_{,i} \quad ,$$

that is the tensor $S_{k\ell m}$ cancels identically.

For static solutions of the field equations, (I-4) reduces to

$$T_{i\beta,\beta} = 0$$

from which it follows that

$$\int \dot{x}_\gamma \; T_{i\beta,\beta} \; d^3x = 0 \quad . \qquad \text{(I-10)}$$

Using $(\dot{x}_\gamma T_{i\beta})_{,\beta} = \delta_{\gamma\beta} T_{i\beta} + \dot{x}_\gamma T_{i\beta,\beta}$ and the four-form of Gauss' theorem, we find that static solutions of (I-1) satisfy the integral relations:

$$\int T_{i\gamma} \; d^3x = \oint \dot{x}_\gamma \; T_{i\beta} ds_\beta \quad . \qquad \text{(I-11)}$$

In particular static solutions for which $\dot{x}_\gamma T_{i\beta} \to 0$ as $\dot{x}_\gamma^{\;2} \to \infty$ satisfy

$$\int T_{i\gamma} \; d^3x = 0 \; .$$

The Lagrangian

$$L = \int \mathcal{L}(A_i, \; A_{i,k}, \; \dot{x}_\ell) \; d^3x$$

of the system is related to its rest mass:[1]

---

1 The "rest mass" M is simply the total energy W of the system, evaluated in the rest frame.

$$M = - \int T_{44} \, d^3x \quad .$$

This relation, as is shown in Appendix B, can be expressed as

$$M = -L - \frac{N}{4\pi} \int \frac{\partial \vec{A}}{\partial t} \cdot \vec{E} \, d^3x - \frac{N}{4\pi} \int \phi \vec{E} \cdot d\vec{s} \quad . \qquad \text{(I-12)}$$

II. PLANE WAVE SOLUTIONS

Although the existence of solutions of the field equations (I-1) corresponding to plane waves is assured by the fact that Maxwell's free field equations (with the fixed gauge $A_k^2 = -1$) are a special case of (I-1), the existence of non-Maxwellian plane wave solutions is also indicated. In order to reduce the field equations to a more tractable form we consider solutions having the particular form

$$\vec{A}(\vec{r},t) = A(\vec{r},t)\vec{e},$$
$$\phi^2 = \gamma^2 A^2 ,$$

where $\vec{e}$ is a unit vector fixed in space and γ is a scalar constant. For such a case, the field equations (in vectorial form) reduce to

$$\vec{e}(\Delta A - \frac{\partial^2 A}{\partial t^2}) - \nabla(\vec{e}\cdot\nabla A + \gamma\frac{\partial A}{\partial t}) = [1 + (1 - \gamma^2)A^2]A\vec{e}$$

and $\hfill$ (II-1)

$$\gamma\ \Delta A + \vec{e}\cdot\nabla\frac{\partial A}{\partial t} = \gamma[1 + (1 - \gamma^2)A^2]A \quad .$$

For plane wave solutions, we assume that

$$A(\vec{r},t) = A(\sigma)$$

where $\sigma = \vec{k}\cdot\vec{r} - \omega t$, $\vec{k}$ and ω being real vector and scalar constants respectively. Equations (II-1) then reduce to

$$[\vec{e}(k^2-\omega^2) + (\omega\gamma-k_e)]\,\frac{d^2A}{d\sigma^2} \;=\; [A + (1-\gamma^2)A^3]\,\vec{e}$$

$$(\gamma k^2-\omega k_e)\,\frac{d^2A}{d\sigma^2} = [A + (1-\gamma^2)A^3]\,\gamma \qquad \text{(II-2)}$$

where $k \equiv |\vec{k}|$ and $k_e = \vec{k}\cdot\vec{e}$. The solutions of (II-2) must also satisfy the generalised Lorentz gauge $\frac{\partial}{\partial x_i}[(1 + A_k{}^2)A_i] = 0$, which takes the form

$$(k_e-\gamma\omega)\,\frac{d}{d\sigma}\,[A + (1 - \gamma^2)A^3] = 0 \; .$$

For nontrivial solutions, this implies that

$$\gamma = \frac{k_e}{\omega} \; ,$$

whereupon the equations (II-2) reduce to the single equation

$$(k^2-\omega^2)\,\frac{d^2A}{d\sigma^2} \;=\; A + (1 - \frac{k_e{}^2}{\omega^2})\,A^3 \; . \qquad \text{(II-3)}$$

The solutions of (II-3) may be expressed in terms of Jacobi elliptic functions.[1] If $k^2 \geq k_e{}^2 > \omega^2$ the general solution of (II-3) may be written as

$$A = \frac{\mu}{\sqrt{1+L^2}}\;\mathrm{sd}\,(\lambda\sigma\sqrt{1+L^2} - \eta,\; K = \frac{L}{\sqrt{1+L^2}}\;) \; ,$$

where λ and η are arbitrary real constants, while

1 Definitions and properties of Jacobi elliptic functions may be found in (14), p. 145.

$$\mu^2 = 2\cdot \frac{k^2-\omega^2}{(k_e/\omega)^2-1}\left(\lambda^2 + \frac{1}{k^2-\omega^2}\right) \quad \text{and} \quad L^2 = 1 + \frac{1}{\lambda^2(k^2-\omega^2)} \quad .$$

Correspondingly,

$$\vec{E} = \mu\lambda\left(\omega\vec{e} - \frac{k_e}{\omega}\vec{k}\right)\frac{\mathrm{cn}(\lambda\sigma\sqrt{1+L^2} - \eta,\ K)}{\mathrm{dn}^2(\lambda\sigma\sqrt{1+L^2} - \eta,\ K)}$$

and

$$\vec{H} = \frac{\vec{k}}{\omega} \times \vec{E} \quad .$$

The relative directions of the various vectors is shown in figure 2.1.

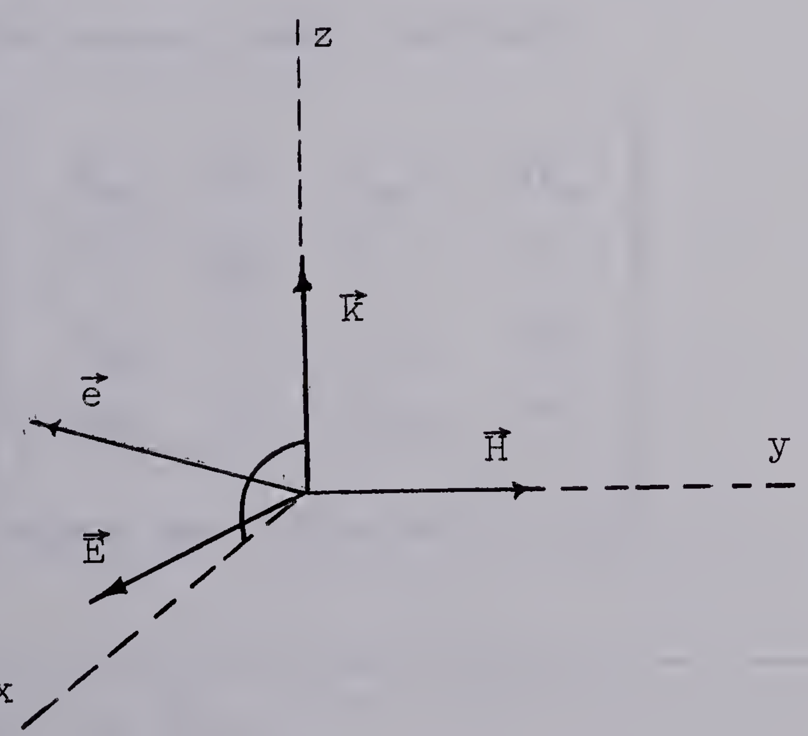

Figure 2.1

The invariants[1] of the field are:

$$\vec{E}\cdot\vec{H} = 0$$

and $\vec{E}^2 - \vec{H}^2 = E^2 \{1 - \frac{k^2}{\omega^2} [1 - \cos^2(\vec{K},\vec{E})]\}$, which can be $>$ or < 0 depending on the value of $\cos(\vec{K},\vec{E})$, the cosine of the angle between the vectors $\vec{K}$ and $\vec{E}$; $E \equiv |\vec{E}|$.

The corresponding charge and current densities are:

$$\rho = \frac{A\gamma}{4\pi} [A^2(\gamma^2-1) - 1]$$

and $\vec{J} = \gamma\rho\vec{e}$,

where $j_k \equiv (\vec{J}, i\rho)$.

The energy-momentum tensor takes the form

$$T_{ij} = \begin{bmatrix} T_{xx} & 0 & T_{xz} & ip_x \\ 0 & T_{yy} & 0 & 0 \\ T_{xz} & 0 & T_{zz} & ip_z \\ ip_x & 0 & ip_z & -W \end{bmatrix}$$

with coordinates chosen as indicated in figure 2.1.

---

1 Reference (12), p. 63.

[1]

$$W = \frac{1}{8\pi}\left[E^2 + \left(\frac{kE}{\omega}\right)^2 - \left(\frac{\vec{k}\cdot\vec{E}}{\omega}\right)^2 + (1+\gamma^2)A^2 + \tfrac{1}{2}(1-\gamma^2)(1+3\gamma^2)A^4 + K\right]$$

$$\vec{p} = \frac{1}{4\pi}\left[E^2\vec{k} - (\vec{E}\cdot\vec{k})\vec{E} + \{1 + (1-\gamma^2)A^2\}\,\gamma A^2\vec{e}\,\right]$$

We note that unlike for Maxwellian plane waves $\vec{E}$, $\vec{H}$ and $\vec{k}$ do not form an orthogonal triad. The electric field always has a component in the longitudinal direction (parallel to $\vec{k}$). In fact if $k^2 = k_e^{\,2}$, $\vec{E}$, $\vec{e}$ and $\vec{k}$ become collinear and $\vec{H} = 0$ so that a purely electric longitudinal plane wave results. Also unlike Maxwellian plane waves, there exist non-null charge and current densities. Furthermore, the current density $\vec{J}$ is generally not parallel to the direction of propagation of the wave.

When $\omega^2 > k^2 \geq k_e^{\,2}$ the situation is analogous to the previous one except that the solutions of (II-3) now take the form

$$A = \mu\ \mathrm{cn}\left(\frac{\lambda}{K}\sigma - \eta,\ K\right)$$

where $\mu^2 = 2(\lambda\omega)^2\,\dfrac{\omega^2-k^2}{\omega^2-k_e^{\,2}}$ and $\dfrac{1}{K^2} = 2 + \dfrac{1}{\lambda^2(\omega^2-k^2)}$.

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1 The constant N which appears in the definition (I-2) of $\mathcal{L}$ is taken to be +1 as for Maxwell's free field equations.

Therefore

$$\vec{E} = \frac{\mu\lambda}{\omega K}\,(k_e\vec{K} - \omega^2\vec{e})\ \mathrm{sn}\,(\tfrac{\lambda}{K}\sigma - \eta, K)\ \mathrm{dn}(\tfrac{\lambda}{K}\sigma - \eta, K)$$

and the invariant

$$E^2 - H^2 = E^2(1 - \frac{k^2}{\omega^2}) + (\frac{\vec{K}\cdot\vec{E}}{\omega})^2$$ is always greater than zero. For the special case $\gamma = 0$ (hence $k_e = 0$) $\vec{E}$, $\vec{H}$ and $\vec{K}$ form an orthogonal triad of vectors; nevertheless there still exists a non-null current density $\vec{J}$ parallel to $\vec{E}$, that is in a transverse direction.

Finally, the case where $k^2 > \omega^2 \geq k_e^2$ does not correspond to any conceivable physical situation, since the solutions of (II-3) are then reciprocals of Jacobi elliptic functions and are therefore periodically singular.[1]

The above solutions are not generalisations of Maxwell's free field solutions in the sense that it is not possible to choose the parameter γ in such a way that (II-1) reduce to Maxwell's free field equations. For such a possibility to arise it would be necessary to introduce some parameter into the expression $1 + A_i^2$ in such a way that $1 + A_i^2 \to 0$ for some particular value (or values) of this parameter. This is impossible if the universal constant $m \neq 0$. Thus the nonlinearity cannot in any sense be considered as a perturbation of the Maxwellian case.

1 Some other nonphysical solutions are discussed briefly in Appendix C.

The field equations (I-1) do not have solutions which would correspond to spherical waves, since this would require that the "argument function" σ take on the form $\sigma = f(r) + g(t)$. The field equations, in spherical polar coordinates[1] contain terms of the form $\frac{\partial^2\phi}{\partial r\partial t} = \frac{df}{dr} \cdot \frac{dg}{dt} \cdot \frac{d^2\phi}{d\sigma^2}$, which (for nontrivial solutions) implies that the product $\frac{df}{dr} \cdot \frac{dg}{dt}$ must be a function of σ only and this is possible only if σ is of the form $\sigma = kr - \omega t$. With such a σ however the equations do not reduce to ordinary equations in σ.

The general case of the field equations, for $\vec{H} \neq 0$, was investigated at some length, but without any success.

For the purpose of investigating possible particle-like (i.e., localised) solutions it is best to use spheroidal coordinates of one sort or another.[2] However, even in the simplest spheroidal coordinates - the spherical polar coordinates - the field equations constitute a very complicated system of four nonlinear partial differential equations. This system of equations is exhibited in Appendix D. The difficulty is compounded by the fact that the generalised Lorentz gauge

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1 The field equations in spherical polar coordinates are given in Appendix D. Throughout this thesis (r, θ, φ) will be used to represent spherical polar coordinates.

2 The more familiar ellipsoidal and spheroidal coordinates are discussed in Reference (16), Chapter 5.

which is inherent in the theory[1] provides very little leeway for simplifying this system of equations. Attempts were made at reducing this system of equations to a more tractable form by considering various special cases, where at most two components of A_i are non-zero and either $\frac{\partial A_i}{\partial \varphi}$ or $\frac{\partial A_i}{\partial \theta} = 0$. In all cases where reduction to reasonably tractable equations was possible, the results were found to be either trivial or unphysical.

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1 See Chapter I, p. 5.

III. STATIC, PARTICLELIKE SOLUTIONS WITH $\vec{H} = 0$.

The field equations (I-1) take on a simple form if the magnetic field strength $\vec{H} = 0$. For the corresponding static case, when $\frac{\partial A_i}{\partial t} = 0$, they reduce to

$$(1 + \vec{A}^2 - \phi^2)\,\vec{A} = 0 \qquad \text{(III-1)}$$

and

$$\Delta\phi = (1 + \vec{A}^2 - \phi^2)\phi\ . \qquad \text{(III-2)}$$

The corresponding Lagrangian (from (I-2)) is [1.]

$$L = -\frac{N}{4\pi}\int [-\tfrac{1}{2}\,(\nabla\phi)^2 + \tfrac{1}{2}(\vec{A}^2-\phi^2) + \tfrac{1}{4}(\vec{A}^2-\phi^2)^2 + K]\;d^3x. \qquad \text{(III-3)}$$

Equation (III-1) implies that $\vec{A} = 0$ or/and $1 + \vec{A}^2-\phi^2 = 0$. If $1 + \vec{A}^2-\phi^2 = 0$ everywhere then equation (III-2) just becomes Laplace's equation which, as is well known, possesses no particlelike solutions (except the trivial solution ϕ = const.). If $\vec{A} = 0$ everywhere, then (III-2) implies that the scalar potential ϕ must satisfy the equation

$$\Delta\phi = \phi - \phi^3. \qquad \text{(III-4)}$$

---

1. The constants N and K are chosen so that the energy is finite and positive, and in agreement with Maxwell's theory where $\vec{A}^2 - \phi^2 + 1 = 0$. See Ref. (7) and Chapters VI and VII.

This particular situation has been considered by Dr. Schiff (7), and the results are summarised in Chapter V. A discrete, countable set of particlelike solutions does exist in this case. As is clear from (III-4), they are asymptotic to zero as $\frac{e^{-r}}{r} Y(\theta,\varphi)$[1] for large r, and therefore correspond to particles of zero charge (since $\int \Delta\phi \; d^3x = 0$).

The most general implication of equation (III-2) is that the condition $1 + \vec{A}^2 - \phi^2 = 0$ hold in some region V of space, while the other condition $\vec{A} = 0$ apply elsewhere. On the boundary S between these regions both conditions must apply: $\vec{A} = 0$ and $\phi = \pm 1$. Furthermore, $\vec{E} = -\nabla\phi$ must exist and be continuous everywhere for a particlelike solution, hence in particular on S. The equation (III-2) then becomes

$$\Delta\phi = \phi - \phi^3 \quad \text{in } V, \qquad \text{(III-5)}$$

$$\Delta\phi = 0 \quad \text{outside } V, \qquad \text{(III-6)}$$

with $\phi = \pm 1$ on the boundary surface S. If V is a closed, simply connected region of space then, choosing the origin of coordinates inside V, the general well behaved solution of (III-6) may be written as

---

1 $Y(\theta,\varphi)$ is some linear combination of the spherical harmonics.

$$\phi = \pm 1 - \alpha + \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{b_{\ell m}}{r^{\ell+1}} Y_{\ell}^{m}(\theta,\varphi) , \qquad \text{(III-7)}$$

α, $b_{\ell m}$ being constants and Y_{ℓ}^{m} the usual spherical harmonics. The boundary surface S is represented by $r = R(\theta,\varphi)$, where

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{b_{\ell m}}{R^{\ell+1}} Y_{\ell}^{m} = \alpha . \qquad \text{(III-8)}$$

The vector potential $\vec{A}$, outside V, is given by $\vec{A}^2 = \phi^2 - 1$; its direction must be radial if $\vec{H}$ is to be zero for any ϕ satisfying Laplace's equation. Inside V, ϕ must be a well behaved solution of (III-5) with $\phi = \pm 1$ and $\frac{\partial\phi}{\partial r} = - \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(\ell+1) b_{\ell m}}{[R(\theta,\varphi)]^{\ell+2}} Y_{\ell}^{m}(\theta,\varphi)$ on S.[1]

The physical interpretation of such solutions, if they exist, is determined by the values of the constants $b_{\ell m}$. Thus, if all $b_{\ell m} = 0$ save $b_{00} = b$, then the solution clearly corresponds to a spherically symmetric charge particle of charge $b(\frac{\hbar c}{g})$.[2] The boundary surface in this case is a

---

1 Note that the continuity of $\frac{\partial\phi}{\partial\theta}$ and $\frac{\partial\phi}{\partial\varphi}$, which are required if $\nabla\phi$ is to be continuous on S, is assured.

2 Since for every solution ϕ of (III-5), $-\phi$ is also a solution, it is clear that such particles occur in oppositely charged pairs.

sphere of radius $R = \frac{b}{\alpha}$, inside which the spherically symmetrical[1] charge density is non-zero. Similarly, if all $b_{\ell m} = 0$ except $b_{00} = b$ and $b_{10} = p$, the solution corresponds to a charged particle with charge and electric dipole moment proportional to b and p respectively. The boundary surface in this case is the spheroid

$$r = R(\mu) = \frac{b}{2\alpha}\left(1 + \sqrt{1 + \frac{4\alpha p \cos\theta}{b^2}}\right) , \quad \text{with}$$

$\left|\frac{4\alpha p}{b^2}\right| < 1.$

The charge densities corresponding to such charged particlelike solutions are given by

$$\begin{aligned} \rho &= -\frac{1}{4\pi}(\phi - \phi^3) \quad \text{in } V \\ &= 0 \qquad\qquad\qquad \text{elsewhere.} \end{aligned} \tag{III-9}$$

Generally speaking, the expression (III-7) represents the electric multipole expansion of the particle's localised charge distribution (III-9).[2]

---

1 It is shown in Appendix E that well behaved solutions of (III-5) which are spherically symmetric on S are such throughout.

2 This does not mean that particlelike solutions corresponding to any values of $b_{\ell m}$ are possible. For example, the scheme does not admit a neutral particle with only a pure electric dipole moment: all $b_{\ell m} = 0$ save $b_{10} = p$, since this implies that the boundary surface S, given by $\alpha r^2 = p \cos\theta$, is not real.

It is possible to consider more complex subdivisions of space into alternating regions of linearity (where $1 + \vec{A}^2 - \phi^2 = 0$ and $\rho = 0$) and nonlinearity (where $\vec{A} = 0$ and $\rho \neq 0$). We consider here only the case of spherical symmetry: assume a subdivision of space into spherical regions by concentric spheres of radii $0 < R_1 < R_2 < \ldots < R_n$, where $\phi(R_i) = \pm 1$, for all $i = 1, 2, \ldots, n$. The innermost region (where $0 \leq r \leq R_1$) is necessarily a "nonlinear" region, since otherwise the only well behaved solutions are the trivial solutions $\phi \equiv \pm 1$.

If $n = 1$, we have the situation discussed above. $n = 2$ implies that

$$\begin{aligned} \Delta\phi &= \phi - \phi^3 \qquad \text{for } 0 \leq r \leq R_1 \text{ and } R_2 \leq r < \infty \\ &= 0 \qquad\qquad \text{for } R_1 \leq r \leq R_2 \end{aligned}$$

with $\phi = \pm 1$ on $r = R_1$ and $\phi = \mp 1$ on $r = R_2$. Well behaved solutions of this system correspond to neutral particles.[1] Such solutions are discussed in Chapter VI.

---

1 As is pointed out in Chapter IV, the only well behaved solutions of (III-4) in $R_2 \leq r < \infty$ are those asymptotic to zero as $\frac{e^{-r}}{r}$.

Similarly, when $n = 3$, well behaved solutions of the system of equations

$$\begin{aligned} \Delta\phi &= \phi - \phi^3 && \text{if } 0 \leq r \leq R_1 \text{ and } R_2 \leq r \leq R_3 \\ &= 0 && \text{if } R_1 \leq r \leq R_2 \text{ and } R_3 \leq r < \infty \,, \end{aligned}$$

with $\phi = \pm 1$ when $r = R_1, R_3$ and $\phi = \mp 1$ when $r = R_2$, correspond to charged particles, since in the outermost region, $R_3 < r$, the well behaved solutions are of the form $a + \frac{b}{r}$. Such solutions are discussed in Chapter VII.

In general, for any $n \geq 0$, well behaved solutions (when they exist) correspond to neutral particles when the outermost region, $R_n \leq r < \infty$, is "nonlinear" and to charged particles when it is "linear".

IV. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF $\Delta\phi = \phi - \phi^3$

It is evident from the discussion of the previous chapter that the existence of neutral and charged particlelike solutions (with $\vec{H} = 0$) is closely tied in with the existence of well behaved solutions of the nonlinear, elliptic partial differential equation

$$\Delta\phi = \phi - \phi^3 \quad {}^{1.} \tag{IV-1}$$

Phase space analysis of the equation for the case of spherical symmetry (15) suggests the existence of a countable infinity of discrete global solutions which are asymptotic to zero as $\frac{e^{-r}}{r}$, together with a continuum of solutions asymptotic to ± 1 as $\frac{\sin(\sqrt{2}\, r + \beta)}{r}$, β being a constant. The existence of the discrete set of solutions asymptotic to zero is suggested also by the fact that equation (IV-1) may be written in the "Schrödinger" form:[2]

$$[-\Delta - \phi^2]\,\phi = -\phi \quad ,$$

---

1. The equation possesses three trivial solutions $\phi = 0, \pm 1$.
2. This was pointed out by Dr. Schiff, (7).

which may be interpreted as the wave equation for the eigenfunctions ϕ of a particle with fixed "energy" -1, moving under the influence of an attractive potential $V(\vec{r}) = -\phi^2$. This point of view suggests the existence of non-spherical eigensolutions in addition to those of spherical symmetry.

The fact that well behaved solutions of (IV-1) do in fact exist is easily demonstrated: For the sake of simplicity we consider solutions of (IV-1) which are axially symmetric $(\frac{\partial\phi}{\partial\varphi} \equiv 0)$.

The substitutions $f(r,\mu) = r\phi(r,\mu)$ and $\mu \equiv \cos\theta$ reduce (IV-1) to

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} P[f] - f + \frac{f^3}{r^2} = 0 \quad , \qquad \text{(IV-2)}$$

where

$$P[f] \equiv \frac{\partial}{\partial\mu} [(1-\mu^2) \frac{\partial f}{\partial\mu}] \quad .$$

Assuming that solutions of (IV-2) are well behaved, let

$$f(r,\mu) = \sum_{\ell=0}^{\infty} \alpha_\ell (\mu)\, r^{\ell+1} \qquad \text{(IV-3)}$$

Then

$$\frac{\partial^2 f}{\partial r^2} = \sum_{\ell=0}^{\infty} \alpha_\ell(\mu)(\ell+1)\ell\; r^{\ell-1} = \sum_{\ell=-1}^{\infty} (\ell+3)(\ell+2)\alpha_{\ell+2}(\mu)r^{\ell+1} ,$$

$$\frac{1}{r^2} P[f] = \sum_{\ell=0}^{\infty} P\,[\alpha_\ell(\mu)\,]\; r^{\ell-1} = \sum_{\ell=-2}^{\infty} P\,[\alpha_{\ell+2}(\mu)]\; r^{\ell+1} \quad \text{and}$$

$$\frac{f^3}{r^2} \quad \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_\ell(\mu)\; \alpha_m(\mu)\; \alpha_n(\mu)\; r^{\ell+m+n+1}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{n=0}^{m} \alpha_{\ell-m}(\mu)\; \alpha_{m-n}(\mu)\; \alpha_n(\mu)\; r^{\ell+1} .$$

Substituting these expressions into (IV-2) and equating the coefficients of the various powers of r, we find that

$$\alpha_0(\mu) = c_0, \text{ an arbitrary constant.}$$

$$P[\alpha_1(\mu)] + 2\cdot 1\; \alpha_1(\mu) = 0$$

$$P\,[\alpha_{\ell+2}(\mu)] + (\ell+3)(\ell+2)\alpha_{\ell+2}(\mu) = \alpha_\ell(\mu) - \sum_{m=0}^{\ell} \sum_{n=0}^{m} \alpha_{\ell-m}(\mu)\alpha_{m-n}(\mu)\; \alpha_n(\mu) , \tag{IV-4}$$

for $\ell = 0, 1, 2, \ldots$

This is a set of linear, but inhomogeneous Legendre equations and the existence of well behaved solutions (for $|\mu| \leq 1$) is

easily established by induction.[1] The general well behaved solution of (IV-4) may be written as

$$\alpha_k(\mu) = c_k P_k(\mu) + \gamma_k(\mu;\ c_o,\ c_1,\ \ldots,\ c_{k-2}) ,$$

where P_k is the Legendre polynomial of order k, c_k an arbitrary constant and γ_k a polynomial of degree k-2 in μ.[2] The general solution for $\alpha_k(\mu)$ thus contains k-1 arbitrary constants.

It is shown in Appendix G that

$$\alpha_\ell(-\mu) = (-1)^\ell \; \alpha_\ell(\mu)$$

hence, as a corollary, the following theorem applies to the solutions:

$$f(r,-\mu) = \pm\, f(r,\mu) \quad \text{if and only if} \quad \alpha_\ell(\mu) = 0 \quad \text{for}$$

$$\text{all} \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} \ell..$$

---

1 This is done in Appendix F.

2 It is a special solution of the complete equation for $\alpha_k(\mu)$. A definition of γ_k is given in Appendix F.

It now remains to consider the radius of convergence of the indicated series solution. We consider the case of spherical symmetry, for which $\frac{\partial f}{\partial \mu} = 0$ or equivalently $c_k = 0$ for all $k > 0$. The coefficients α_k are now constants (which depend on c_o) and as such are trivially even functions of μ, so that $\alpha_\ell = 0$ for all odd ℓ. For even ℓ, we obtain from (IV-4)

$$\alpha_o = c_o,$$

$$(\ell+3)(\ell+2)\,\alpha_{\ell+2} = \alpha_\ell - \sum_{m=0}^{\ell} \sum_{n=0}^{m} \alpha_{\ell-m}\,\alpha_{m-n}\alpha_n. \qquad \text{(IV-5)}$$

Using the ratio test, it is easily seen by induction that the series converges uniformly to an analytic function in the region $0 \leq r \leq R(c_o)$ for every finite c_o[1], although R decreases with increasing $|c_o|$[2]. In the region $\frac{R}{2} \leq r < R'$, where R' is arbitrarily greater than R, the equation

$$\frac{d^2f}{dr^2} - f + \frac{f^3}{r^2} = 0 \qquad \text{(IV-6)}$$

now satisfies a Lipshitz condition, hence Picard's method may be used to establish the existence of a unique solution.[3]

---

1 This is established in Appendix H.

2 Some explicit values of f(r) as computed from the series have been worked out for various α_o. They are listed in Appendix J.

3 (14), p. 83, 189.

Thus for the case of spherical symmetry, (IV-1) has a unique global solution for each value of $\phi(r=0)$. All these solutions have the property that $\frac{d\phi}{dr} = 0$ at $r = 0$ and $\phi(-r) = \phi(r)$. Furthermore phase space analysis of the equation (IV-6) (Reference 15) implies that all these solutions are finite everywhere.

For the more general case of axial symmetry, it is similarly seen that the series solution (IV-3) converges in a finite region about the origin. The general series solution contains an infinite set of arbitrary constants c_k, for $k = 0, 1, 2, \ldots$ It is not obvious however that these constants may be so chosen that the series converge globally to nontrivial functions.

The series solutions exhibited here are useful only for determining the behaviour of the solution in the vicinity of the origin. It is impossible to infer from them how global solutions of (IV-1) behave for large values of r. For the case of spherical symmetry this information may be obtained with the aid of phase space analysis, while for more general cases it is not even obvious that global solutions exist.

The problem of determining charged particlelike solutions takes on the form of a Cauchy problem for the nonlinear equation, in that well behaved solutions of (IV-1) are sought in a closed region V of space, with $\phi = \pm 1$ and

$\frac{\partial\phi}{\partial n}$ given everywhere on the bounding surface S. Since the equation (IV-1) is elliptic one might surmise that the problem is poorly posed (overspecified). In fact the boundary value (Dirichlet) problem is also poorly posed (underspecified) in that if, for example, S is taken (arbitrarily) to be the sphere of radius R = 2.3, then there exists a discrete set of solutions (in all probability an infinity of them, corresponding to

$$r^2 \frac{\partial\phi}{\partial r} = 0,\ .123,\ 2.706,\ 6.279,\ 12.138,\ 20.161,\ 29.591,\ \text{etc.},$$

on $r = 2.3$.[1]

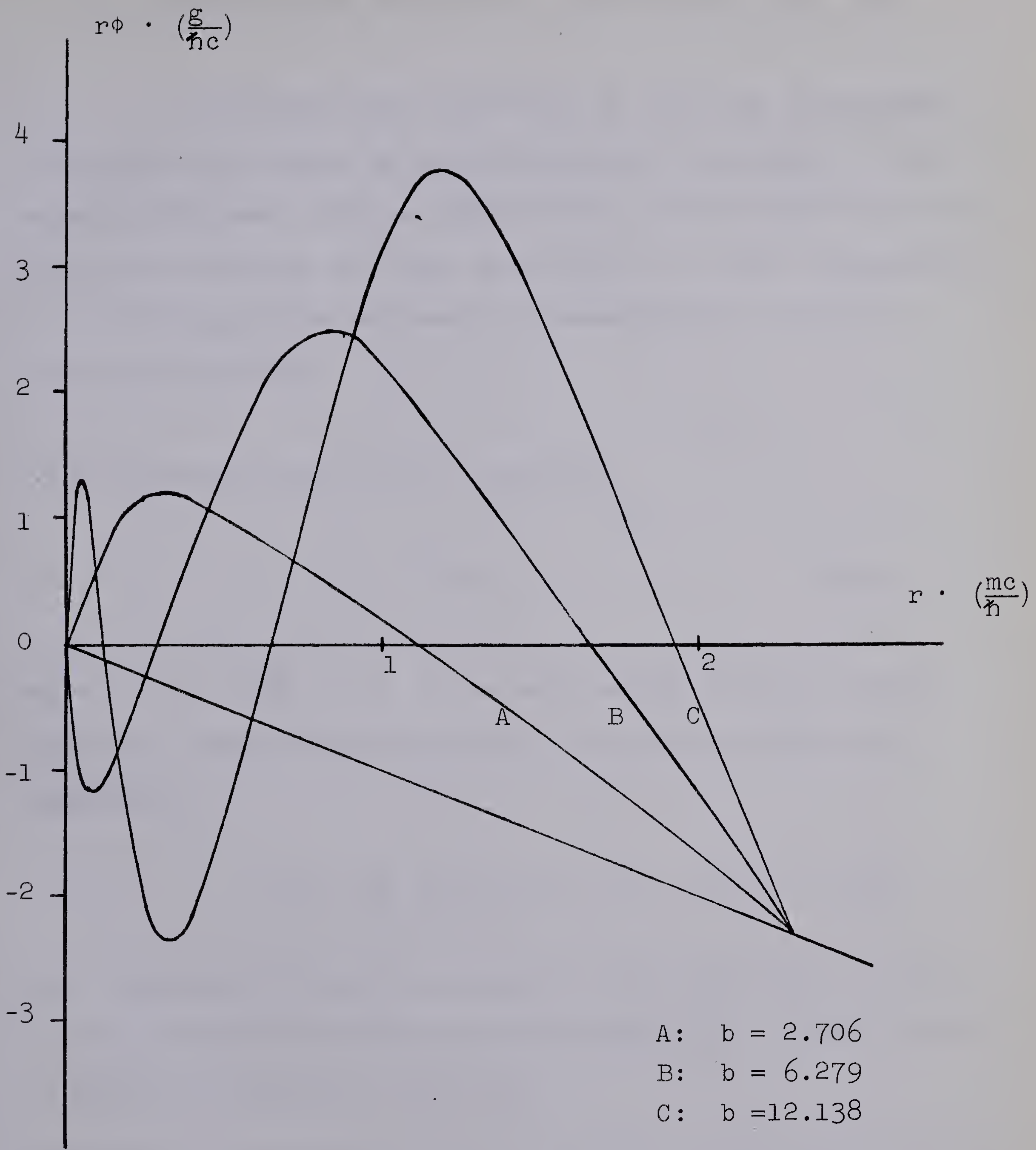
As already remarked the situation here resembles most closely (at least for the spherically symm. case) the eigenvalue problem for the Schrödinger equation, rather than either the Cauchy or Dirichlet problems. Unlike the Schrödinger problem, however, the eigenvalue does not appear explicitly in the equation but is rather contained in the boundary conditions.

For nonspherical charged particles, however, the boundary conditions correspond to Cauchy data, hence the existence of the "interior" solution ($r \epsilon V$) is rather obscure.[2]

---

1 The first few (numerically obtained) solutions are plotted in fig. 4.1.

2 As is the concept of $\frac{\partial\phi}{\partial n}$ corresponding to an "eigenvalue" since $\frac{\partial\phi}{\partial n}$ is no longer a constant on S.



$r\phi \cdot (\frac{g}{\hbar c})$
4
3
2
1
0
-1
-2
-3
$r \cdot (\frac{mc}{\hbar})$
1
2
A
B
C
A: b = 2.706
B: b = 6.279
C: b =12.138

Figure 4.1

V. APPROXIMATION METHODS FOR PARTICLELIKE SOLUTIONS

Nonlinear field equations of the type considered here generally cannot be solved explicitly by means of the usual techniques; various approximation methods must be used. One of the simplest and most successful of these, especially for obtaining approximations to eigensolutions, is the variational method:

Consider field equations of the form

$$\Delta\phi = F'(\phi) \qquad \text{(V-1)}$$

where $F'(\phi) \equiv \frac{dF}{d\phi}$ and F is some simple function of F.[1] Since well behaved solutions of (V-1) also extremise the Lagrangian

$$L(\phi) = \frac{N}{4\pi}\int[\frac{1}{2}(\nabla\phi)^2 + F(\phi)]\, d^3x, \qquad \text{(V-2)}$$

this suggests that approximations to the solutions of (V-1) can be obtained variationally by extremising L with respect to suitable comparison functions.

---

1 For example, if $F'(\phi) = \phi - \phi^3$ in some localised region V of space and zero elsewhere, then (V-1) is identical with (III-5).

If Φ is a solution of (V-1) and $\psi = \Phi + u$ some suitable variational trial function[2] (u being an arbitrary but small perturbation), consider the nature of the bound $L(\psi)$ to $L(\Phi)$:

$$L(\psi) = L(\Phi) + \frac{1}{4\pi} \int [\nabla\Phi\cdot\nabla u + u\, F'(\Phi)]d^3x$$
$$+ \frac{1}{4\pi} \int [\tfrac{1}{2}(\nabla u)^2 + \tfrac{1}{2}\, u^2\, F''(\Phi)]d^3x \qquad \text{(V-3)}$$
$$+ \frac{1}{4\pi} \sum_{\ell=3}^{\infty} \int \frac{u^\ell}{\ell!}\, F^{(\ell)}(\Phi) d^3x \quad .$$

Using $\nabla\cdot(u\nabla\Phi) = \nabla u\cdot\nabla\Phi + u\,\Delta\Phi$ and Gauss' theorem:

$$\int[\nabla\Phi\cdot\nabla u + u\, F'(\Phi)]d^3x = \oint u\nabla\Phi\cdot d\vec{s} + \int[-\Delta\Phi + F'(\Phi)]u\; d^3x \quad .$$

Since u is arbitrary this implies that L is extremised if and only if Φ is a solution of (V-1).[1]

---

1 Note that for solutions which do not vanish at infinity the indicated surface integral does not in general vanish and then the variational principle which is equivalent to the field equations is
$\delta L - \frac{1}{4\pi}\oint \delta\Phi\; \nabla\Phi\cdot d\vec{s} = 0$ rather than $\delta L = 0$. For example, this is the case for charged particlelike solutions discussed in Chapter VII.

2 Throughout this thesis the terms "variational trial function" and "variational comparison function" will be used interchangeably.

If u is sufficiently small, the nature of the variational bound is determined by the second order term

$$\delta^2 L = \frac{1}{4\pi} \int [\tfrac{1}{2}(\nabla u)^2 + \tfrac{1}{2} u^2 F''(\phi)]d^3x. \qquad \text{(V-3a)}$$

Since $F''(\phi)$ is in general indefinite, so presumably, is the variational bound $L(\psi)$ to $L(\phi)$. Suppose, however, that the variational comparison functions ψ are restricted to those which satisfy the integral relation

$$\int [(\nabla\psi)^2 + 6\, F(\psi)]d^3x^{1.} = 0 \quad , \qquad \text{(V-4)}$$

which relation is also satisfied by the solutions of (V-1). Replacing ψ by $\phi + u$ in (V-4), we get, to second order in u

$$\int \tfrac{1}{2}\, u^2 F''(\phi)d^3x = - \int [\tfrac{1}{3}\nabla\phi\cdot\nabla u + \tfrac{1}{6}(\nabla u)^2 + u\, F'(\phi)]d^3x \ .$$

Thus, to second order in u (V-3) becomes

$$L(\psi) = L(\phi) + \frac{1}{4\pi}\int \frac{2}{3}\, \nabla\phi\cdot\nabla u\ d^3x + \frac{1}{4\pi}\int \frac{1}{3}(\nabla u)^2 d^3x. \qquad \text{(V-5)}$$

Since ϕ is a solution of (V-1) it extremises L hence the first order term in (V-5) vanishes, while the second order

---

1. This is equivalent to requiring that the Lagrangian be an extremum with respect to a radial scale parameter of an arbitrary comparison function ψ, (p. 40).

term is now positive definite. Thus, if the variational comparison function ψ is chosen to satisfy (V-4) then $L(\psi)$ provides an upper bound to $L(\Phi)$ provided $|\psi - \Phi|$ is sufficiently small. This differs significantly from the usual linear eigenvalue problem in that no other conditions (such as the subsidiary orthogonality condition in the linear problem) are necessary to ensure upper bounds to excited states.

The condition (V-4) imposed here on the comparison function ψ is possibly not the only one which, by itself, can assure that $L(\psi)$ give an upper bound to $L(\Phi)$ for $|\psi - \Phi|$ sufficiently small. It is clear that any condition satisfied by the exact solutions of (V-1) may be used, provided it may be utilised to eliminate the generally indefinite term $\int F''(\Phi)u^2\, d^3x$ from $\delta^2 L$,[1.] and reduce the latter to a positive definite form.

Note that in general the integral relation (V-6) below (which is equivalent to extremising L with respect to the amplitude of ψ) does not by itself assure that $L(\psi)$ is an upper bound to $L(\Phi)$.[2.]

1. It may happen that $F''(\Phi)$ is a positive definite quantity, in which case $L(\psi)$ will give an upper bound to $L(\Phi)$ for any ψ such that $|\psi - \Phi|$ is small. This is not the case for $F(\Phi) = \Phi - \Phi^3$.

2. This is because when (V-6) is solved (to second order in u) for $\int F''(\Phi)u^2\, d^3x$ one of the terms which arises, namely $-1/2 \int u^2 \Phi F'''(\Phi) d^3x$, is also indefinite in general. However, as pointed out by Dr. Schiff, for the case $F'(\Phi) = \Phi - \Phi^3$ it is positive definite.

In using the variational principle to obtain approximations to the solutions of (V-1), the variational Lagrangian may be extremised with respect to variable amplitude and scale parameters without otherwise specifying the nature of the comparison function.

Let $\psi(x,y,z) = A\, f(\alpha x,\ \beta y,\ \gamma z,\ \delta_i)$

where $A, (\alpha,\beta,\gamma),\ \delta_i$ $(i = 1,\ 2,\ \ldots)$ are amplitude, scale and other parameters while f is some suitable and integrable[1] comparison function. For $L(\psi)$ to be an extremum with respect to A we require that

$$\frac{\partial L}{\partial A} = \frac{1}{4\pi} \int [\nabla\psi \cdot \frac{\partial}{\partial A} (\nabla\psi) + \frac{\partial F}{\partial A}] d^3x = 0$$

Since $\frac{\partial}{\partial A} (\nabla\psi) = \nabla(\frac{\partial\psi}{\partial A}) = \frac{1}{A} \nabla\psi$ and $\frac{\partial F}{\partial A} = F'(\psi) \frac{\partial\psi}{\partial A} = F'(\psi) \frac{\psi}{A}$

The above condition reduces to

$$\int [(\nabla\psi)^2 + \psi F'(\psi)]\ d^3x = 0. \qquad \text{(V-6)}$$

---

1 "Integrability" here implies that all indicated integrals involving ψ (hence f) exist.

The nature of this extremum is determined by the value of $\frac{\partial^2 L}{\partial A^2}$ at the "extremal" value of A. Noting that $\frac{\partial^2 \psi}{\partial A^2} = 0$,

$$\frac{\partial^2 L}{\partial A^2} = \frac{1}{4\pi A^2} \int [(\nabla\psi)^2 + \psi^2 F''(\psi)] \, d^3x \; ,$$

hence at the extremum (where (V-6) holds):

$$\left.\frac{\partial^2 L}{\partial A^2}\right|_{\text{ext.}} = \frac{1}{4\pi A^2} \int [\psi^2 F''(\psi) - \psi F'(\psi)] \, d^3x \; .$$

Thus, whether the extremum is a maximum or a minimum (or neither) depends on the explicit form of $F(\psi)$. In the particular case $F'(\psi) = \psi - \psi^3$

$$\left.\frac{\partial^2 L}{\partial A^2}\right|_{\text{ext.}} = -\frac{2}{4\pi A^2} \int \psi^4 \, d^3x < 0,$$

so that L is a maximum with respect to A.

To extremise L with respect to scale parameters, consider the transformation $x = \frac{x'}{\alpha}$, whereupon the Lagrangian may be written (with $\beta = \gamma = 1$)

$$L = \frac{1}{4\pi} \int [\frac{1}{2} (\frac{\partial\psi(x',y,z)}{\partial x})^2 + \frac{1}{2} (\frac{\partial\psi}{\partial y})^2 + \frac{1}{2}(\frac{\partial\psi}{\partial z})^2 + F(\psi)]dx\ dy\ dz$$

$$= \frac{1}{4\pi} \int [\frac{\alpha}{2} (\frac{\partial\psi}{\partial x'})^2 + \frac{1}{2\alpha} (\frac{\partial\psi}{\partial y})^2 + \frac{1}{2\alpha} (\frac{\partial\psi}{\partial z})^2 + \frac{F(\psi)}{\alpha}]dx'\ dy\ dz$$

Therefore

$$\frac{\partial L}{\partial\alpha} = \frac{1}{4\pi} \int [\frac{1}{2} (\frac{\partial\psi}{\partial x'})^2 - \frac{1}{2\alpha^2}(\frac{\partial\psi}{\partial y})^2 - \frac{1}{2\alpha^2} (\frac{\partial\psi}{\partial z})^2 - \frac{1}{\alpha^2} F(\psi)]dx'dy\ dz = 0$$

implies (after performing the inverse transformation $x' = \alpha x$):

$$2 \int (\frac{\partial\psi}{\partial x})^2\ d^3x = \int [(\nabla\psi)^2 + 2\ F(\psi)]\ d^3x \quad .$$

Analogous results hold for y and z; hence

$$\int (\frac{\partial\psi}{\partial x})^2\ d^3x = \int (\frac{\partial\psi}{\partial y})^2\ d^3x = \int (\frac{\partial\psi}{\partial z})^2\ d^3x = \int [\frac{1}{2}(\nabla\psi)^2 + F(\psi)]d^3x \tag{V-7}$$

if L is to be an extremum with respect to scale parameters. Relation (V-4) follows immediately from (V-7).

As already mentioned the integral relation (V-4) is also obtainable by extremising L with respect to a radial scale parameter. This particular extremum is actually a maximum for any $F(\psi)$, which implies that integrable (hence particlelike) solutions of (V-1) are unstable with

respect to a uniform radial "stretching" of the particle.[1]

Finally we note that the integral relations (V-4), (V-6) and (V-7) must be satisfied, in particular, by integrable solutions of (V-1).

In using the variational method the usual procedure is to choose a comparison function which is explicit in its dependence on the independent variables but contains also variable parameters. For example a comparison function for a possible odd-parity eigensolution of (III-4) having axial symmetry, might be

$$\phi = A[r\, e^{-\alpha r}\, P_1(\mu) + b\, r^3\, e^{-\beta r}\, P_3(\gamma\mu)], \qquad \text{(V-8)}$$

where P_1 and P_3 are Legendre polynomials, while A, α, β, γ and b are variational parameters. Substitution of (V-8) into the Lagrangian then yields

$$L = L(A, \alpha, \beta, \gamma, b)$$

which can then be extremised with respect to the parameters, "optimal" values of which are thus obtained.

---

1 This was pointed out by R.H. Hobart (Proc. Phys. Soc./ London/ 82, 201 (1963)) for spherically symmetric solutions and by Derrick (18) for general integrable solutions of (V-1).

It is possible, however, to choose comparison functions for which the dependence on one (or more) of the independent variables is not specified. To illustrate the point, consider a variational comparison function of the form

$$\phi = e^{-\alpha r}\, Y(\mu), \tag{V-9}$$

where α is a variable parameter and $Y(\mu)$ is some function to be determined. The Lagrangian then becomes

$$L(Y;\alpha) = \int_{-1}^{1} \mathcal{L}\,(Y, Y', \mu; \alpha)\; d\mu \quad .$$

The resulting Euler-Lagrange equation will yield an ordinary differential equation which may be solved (numerically, if necessary) for the optimal form of $Y(\mu)$ and hence L. It should be noted that even this "optimised" L will (in general) still contain variable parameters (α in the example above) and will therefore have to be extremised with respect to them.

Approximations to particlelike solutions of (V-1) can also be obtained numerically. In the case of spherical symmetry (when (V-1) reduces to an ordinary differential equation) a number of methods exist for obtaining numerical solutions,[1] of which the Runge-Kutta and finite difference

1 See, for example, Fox (19), Chapter I.

methods are best known. The latter method has been used to obtain all numerical solutions quoted throughout this thesis (except where stated otherwise).[1] The finite difference analogue to (IV-6) is

$$\frac{f(r+\delta) + f(r-\delta) - 2f(r)}{\delta^2} - f(r) + \frac{f^3(r)}{r^2} = 0, \quad \text{(V-10)} ,$$

where δ is a small increment in r. In replacing (IV-6) by (V-10) terms of the order of $\delta^2 \frac{d^4 f}{dr^4}$ and higher are neglected. Approximate particlelike solutions of (IV-6) may then be obtained by iteratively solving (V-10). For example for neutral particles, the asymptotic form of the solution (for r greater than some arbitrary, large R) is $A \frac{e^{-r}}{r}$. Values of the solution for r less than R can then be computed from (V-10) for various A. Actual approximate solutions of (IV-6) then correspond to those values of the "asymptotic amplitude" A for which the behaviour of the numerically computed f(r) near the origin, is of the required form.[2] The iteration can also be performed in

---

1 All ordinary differential equations considered here are reducible to the form $\frac{d^2 y}{dx^2} = f(x,y)$ which, according to Fox (19, p. 4) is most easily and accurately solved by the finite difference technique.

2 The form that the particlelike solutions of (IV-6) must have in the vicinity of the origin was discussed in Chapter IV ((IV-3) and (IV-5)).

the reverse order (starting near $r = 0$) as was done (for neutral particles) by Teshima (8).

A similar procedure may (hopefully) be used to obtain numerical approximations to non-spherical particle-like solutions, if such exist. The field equation (III-4), in spherical polar coordinates, is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} - \phi + \phi^3 = 0. \qquad \text{(V-11)}$$

The behaviour of all particlelike solutions in the vicinity of the origin was determined in the previous chapter ((IV-3) and (IV-4)).

For charged particles the behaviour of solutions in the "linear" region (where $\vec{A}^2 - \phi^2 - 1 = 0$) is given by (III-7). For neutral particlelike solutions, the asymptotic form for large r is (from (V-11), with $|\phi| \ll 1$)

$$\phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m}\, k_\ell(r)\, Y_\ell^m(\theta,\varphi)$$

where the Y_ℓ^m are the usual spherical harmonics, while $k_\ell(r) = -i^\ell h_\ell^{(1)}(ir)$, $h_\ell(x)$ being spherical Hankel functions of the first kind ((17, p. 539)). Since

$k_\ell(r) \to \frac{e^{-r}}{r}$ as $r \to \infty$ (for all ℓ) then, for large r, solutions of (V-11) which are asymptotic to zero are of the form $\phi = \frac{e^{-r}}{r} Y(\theta,\varphi)$, where $Y(\theta,\varphi)$ is some linear combination of the spherical harmonics. In any case the behaviour of particlelike solutions for large r is readily determined. Thus, if odd-parity neutral particlelike solutions of (V-11), having axial symmetry are sought then, for r greater than some arbitrary but large R the asymptotic form

$\phi = \sum_{\ell=0}^{\infty} A_\ell k_\ell(r)\, P_{2\ell+1}(\cos\theta)$ may be used. For $r < R$ and $0 \leq \theta \leq \pi$, we replace (V-11) by its finite difference analogue. First however, since the coefficient of $\frac{\partial\phi}{\partial\theta}$ in (V-11) is singular at $\theta = 0$, it is useful to make the transformation $\mu = \cos\theta$ (and $f = r\phi$), whereupon (V-11) becomes

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial f}{\partial\mu}\right] - f + \frac{f^3}{r^2} = 0.^{1} \qquad \text{(V-12)}$$

Its finite difference analogue is

---

1 In analogy to the transformation $f = r\phi$, there exists a further transformation $f = \frac{f'}{\sqrt{\sin\theta}}$ which eliminates the first partial with respect to θ (or μ) from (V-12). However, since this transformation brings in singular coefficients (at $\theta = 0$) into the resulting equation it is not desirable to carry it through (at least if the equation is to be solved numerically).

$$\frac{f(r+\delta,\mu) + f(r-\delta,\mu) - 2f(r,\mu)}{\delta^2} + \frac{1-\mu^2}{r^2}\left\{\frac{f(r,\mu+\Delta)+f(r,\mu-\Delta)-2f(r,\mu)}{\Delta^2}\right\}$$

$$- \frac{2\mu}{r^2}\left\{\frac{f(r,\mu+\Delta)-f(r,\mu-\Delta)}{2\Delta}\right\} - f(r,\mu) + \frac{f^3(r,\mu)}{r^2} = 0 \,, \tag{V-13}$$ [1.]

with terms of the order of $\delta^2 \frac{\partial^4 f}{\partial r^4}$, $\frac{\Delta^2}{r^2}\frac{\partial^4 f}{\partial \mu^4}$, $\frac{\Delta^2}{r^2}\frac{\partial^3 f}{\partial \mu^4}$ and smaller being neglected (δ and Δ are small increments in r and μ respectively). Thus (V-13) might be used to compute values of f for $r < R$ and $|\mu| \leq 1$. A solution would then correspond to those values of the constants A_ℓ for which the behaviour of f near the origin is of the form required. Clearly, this procedure for nonspherical states is much more cumbersome since the asymptotic form of the solution now is indeterminate to within an arbitrary analytic function

$$P(\mu) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(\mu).$$ [2]

---

1 At $\mu = \pm 1$ "forward" and "backward" difference approximations (rather than central) must be used for $\frac{\partial f}{\partial \mu}$ and $\frac{\partial^2 f}{\partial \mu^2}$, since f is undefined for $|\mu| > 1$.

2 The numerical methods described here were used successfully (as a trial) to reproduce the known s and p state eigensolutions of the Schrödinger equation with the potentials $V(r) = -\frac{1}{r}$ and $-e^{-r}$.

VI. NEUTRAL PARTICLES

Neutral particlelike solutions of the static field equations (III-4) were obtained variationally by Betts, Schiff and Strickfadden (10) and numerically (for the case of spherical symmetry) by Teshima (8). In agreement with the implication of phase space analysis (15), numerical solution of (III-4) yields a discrete set $\{\phi_i\}$ of neutral particle-like solutions asymptotic to zero as $A_i \frac{e^{-r}}{r}$. The first four are plotted in figures 6.1 to 6.4.

As shown by Betts et al (10), these eigensolutions satisfy the "orthogonality" relation

$$\int \phi_i^{\,3} \, \phi_j \, d^3x = \int \phi_j^{\,3} \phi_i \, d^3x \qquad \text{(VI-1)}$$

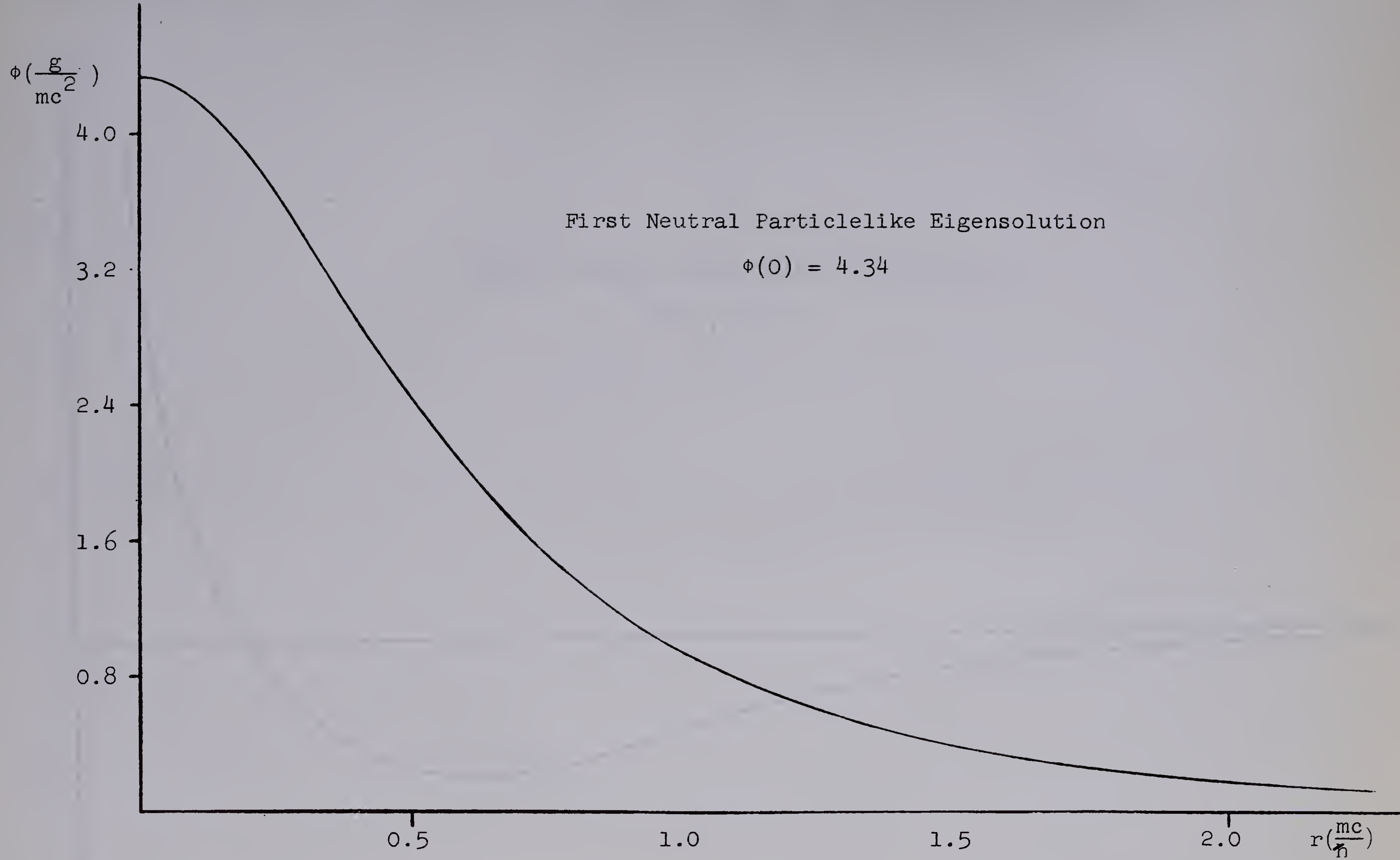
Since $\vec{A} = 0$, the spin[1] of the particles represented by these solutions is zero, as expected. Also, since $\vec{H} = 0$, the Poynting vector $\vec{S} = \frac{1}{4\pi} \vec{E} \times \vec{H}$ is zero, hence so is the angular momentum $J_{\ell m}$ (or $L_{\ell m}$).[2] The Lagrangian of the system is, according to (III-3),

$$L = - \frac{1}{4\pi} \int [\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2} \phi^2 - \tfrac{1}{4} \phi^4] \, d^3x \qquad \text{(VI-2)}$$

---

1 Recall that the "spin density" is proportional to $\vec{A} \times \vec{E}$.

2 These were defined in Chapter I.



First Neutral Particlelike Eigensolution
$\phi(0) = 4.34$
$\phi(\frac{g}{mc^2})$
4.0
3.2
2.4
1.6
0.8
0.5
1.0
1.5
2.0
$r(\frac{mc}{\hbar})$

Fig. 6.1

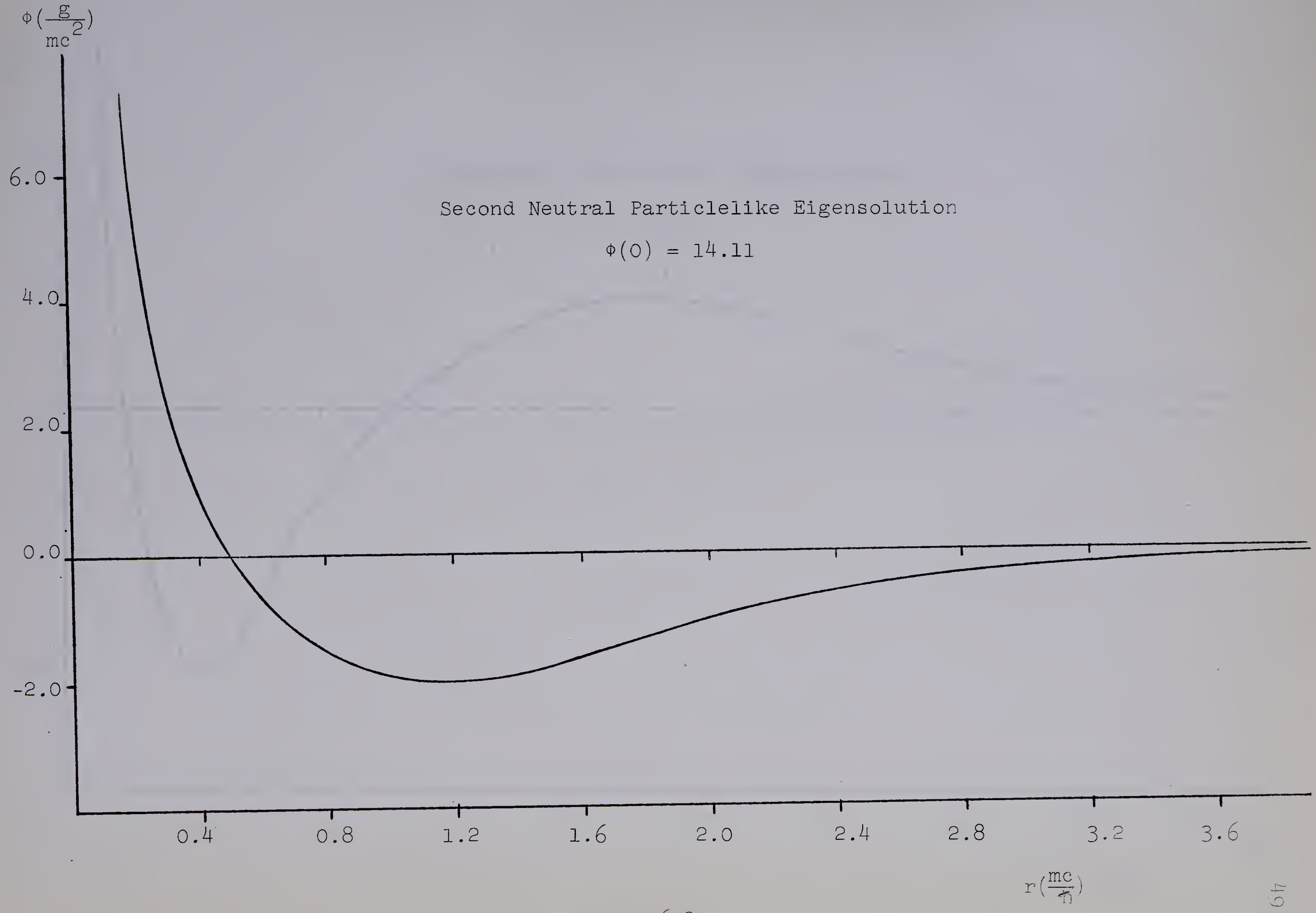
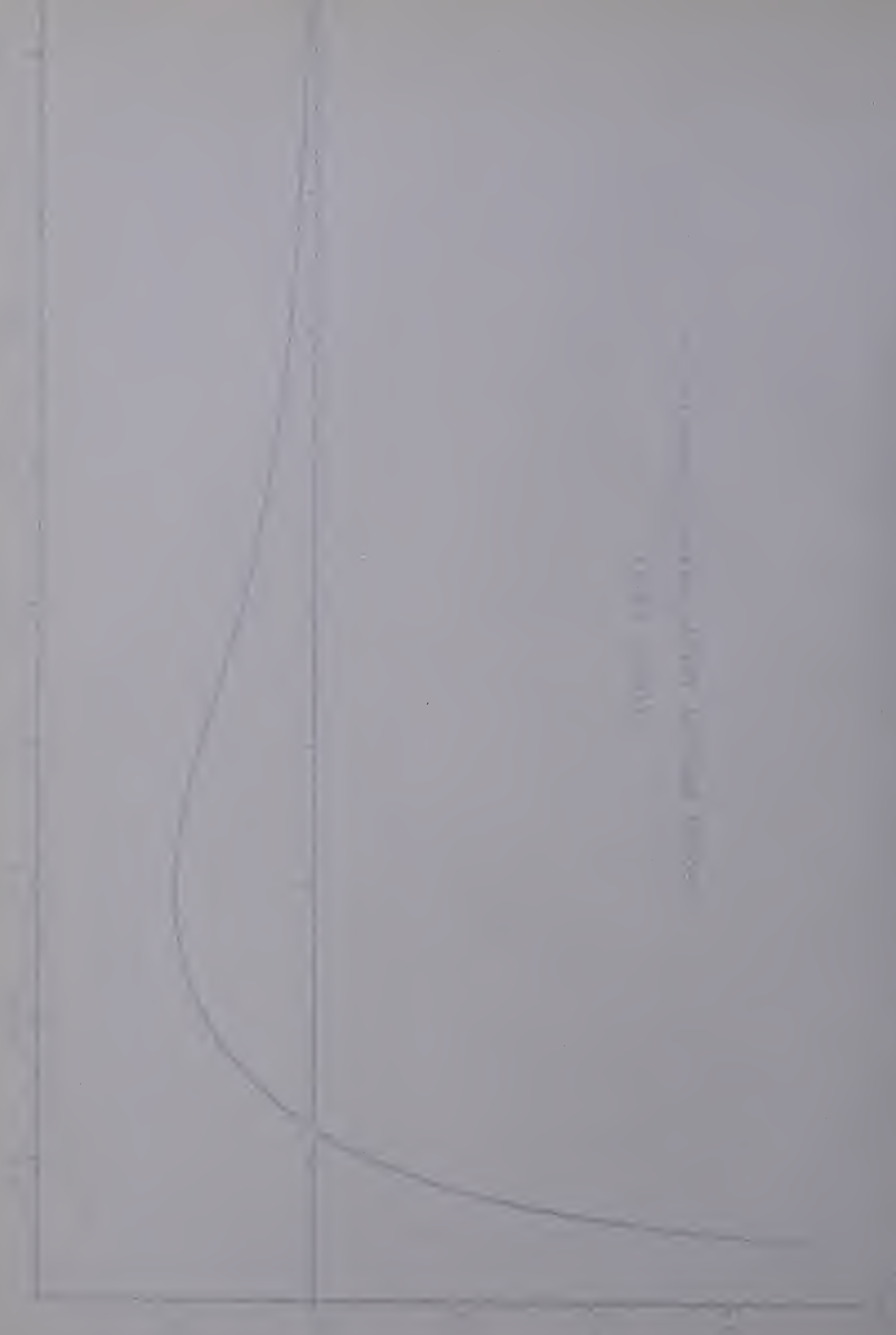
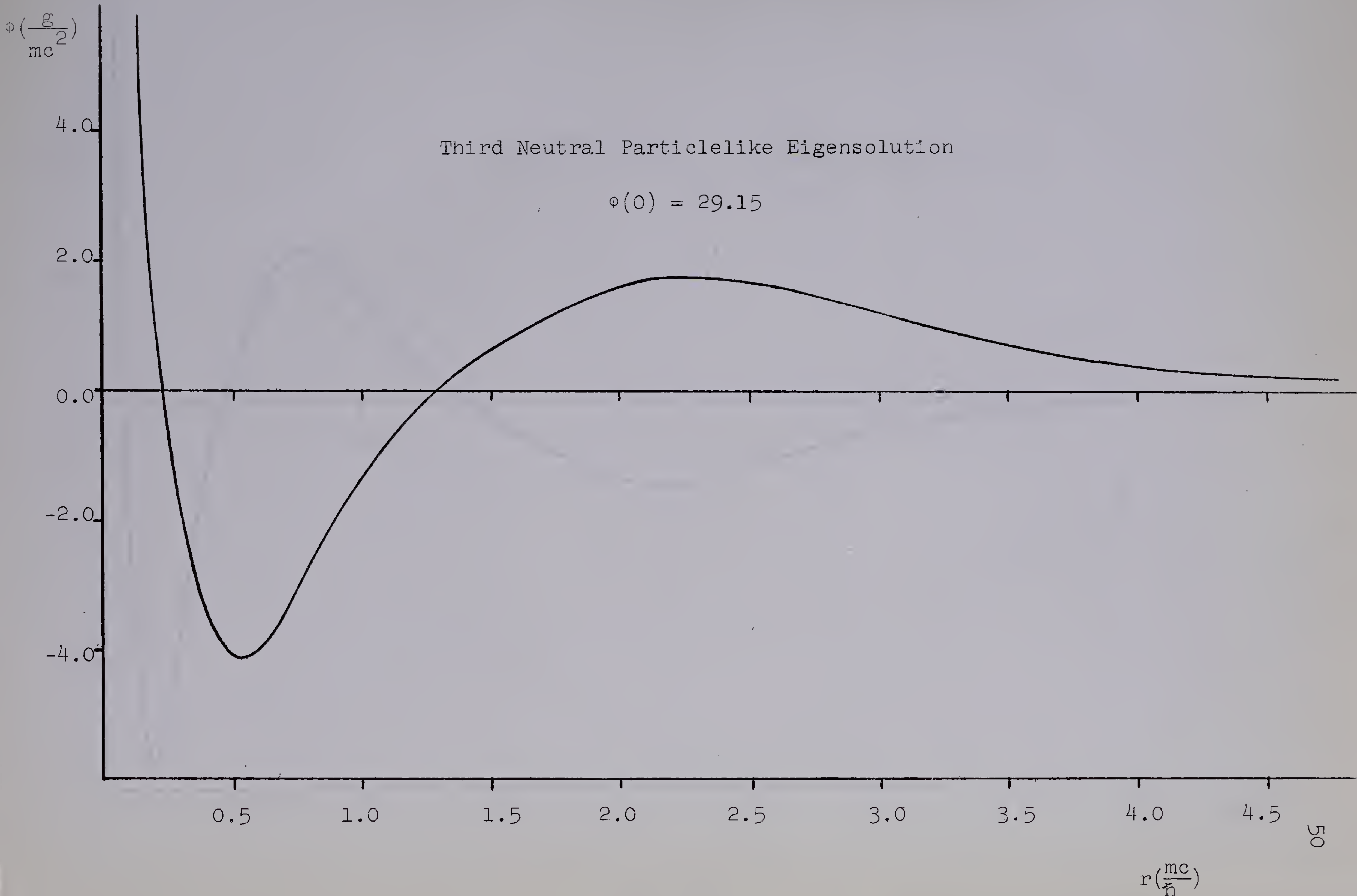

$\phi(\frac{g}{mc^2})$
6.0
4.0
2.0
0.0
-2.0
Second Neutral Particlelike Eigensolution
$\phi(0) = 14.11$
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
$r(\frac{mc}{\hbar})$

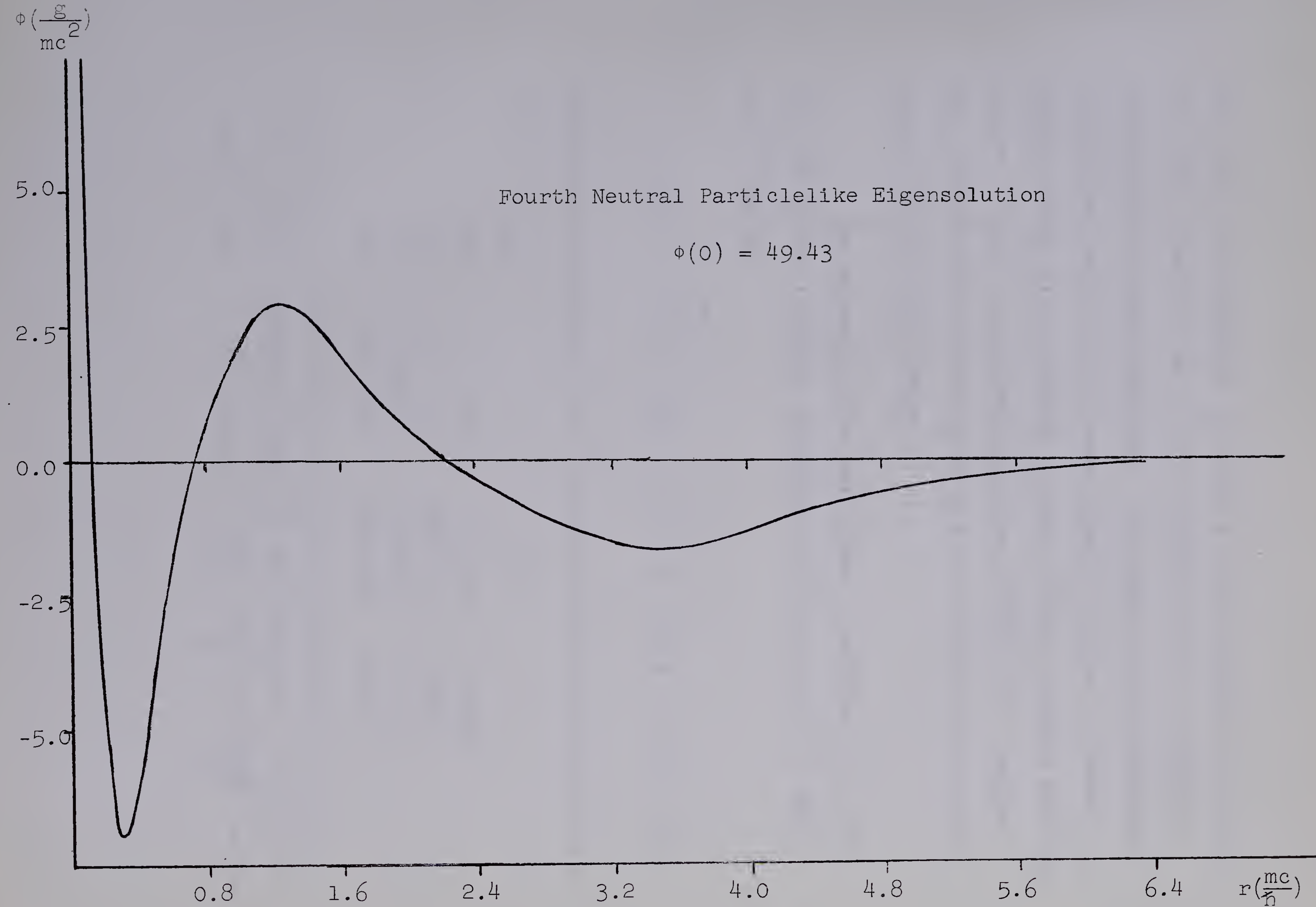
Fig. 6.2





$\phi(\frac{g}{mc^2})$
4.0
2.0
0.0
-2.0
-4.0
Third Neutral Particlelike Eigensolution
$\phi(0) = 29.15$
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
$r(\frac{mc}{\hbar})$

Fig. 6.3



Fourth Neutral Particlelike Eigensolution
ϕ(0) = 49.43
ϕ(g/mc²)
5.0
2.5
0.0
-2.5
-5.0
0.8
1.6
2.4
3.2
4.0
4.8
5.6
6.4
r(mc/ħ)

Fig. 6.4

Note that the constants N and K which appear in (III-3) have been chosen to be -1 and 0 respectively, to assure a finite, positive mass spectrum. The sign of the Lagrangian for neutral particles is thus opposite to that of Maxwell's theory, hence also opposite to that for the charged particle system. This, as pointed out by Dr. Schiff (7), implies that charged and neutral particles are distinctly separate systems within this scheme.

The masses of the particles, according to (I-12) (and using the integral relations (V-4) and (V-6)), may be written as

$$M = \frac{1}{4\pi} \int \phi^2 d^3x = \frac{1}{16\pi} \int \phi^4\, d^3x = \frac{1}{12\ \pi} \int (\nabla\phi)^2\, d^3x.$$

These values for the first few neutral particlelike solutions are:[1]

(n)	A	$\phi(0)$	M
(0)	$\pm$ 2.713	$\pm$ 4.34	1.503
(1)	$\pm$ 17.11	$\mp$14.11	9.46
(2)	$\pm$ 83.87	$\pm$29.15	28.71
(3)	$\pm$375.0	$\mp$49.43	63.97

---

1 (n) is the number of nodes of the solution. $A = \lim_{r\to\infty} re^{r}\phi$. $A, \phi(0)$ and M are in units of $\frac{\hbar c}{g}$, $\frac{mc^2}{g}$ and $\frac{mc^3\hbar}{g^2}$ respectively.

It is reasonable to assume that there exist also solutions of (III-4) which are not spherically symmetric.[1] Variational approximations to a possible non-spherical eigensolution of the form $\Phi(r,\mu) = -\Phi(r,-\mu)$, that is an axially symmetric solution with a nodal plane, were obtained by Betts, Schiff and Strickfadden (10) using the method of parameter variation.[2] Their results, for the lowest (in energy) odd parity state, are reproduced in table VI-A. They seem to suggest the existence of an odd parity, eigensolution corresponding to a neutral particle of rest mass $\lesssim 3.96 \frac{mc^3\hbar}{g^2}$.

The variation of function method, described in Chapter V, was used by this author to obtain approximations to possible odd parity eigenstates. This method requires quite a bit more numerical computation than parameter variation. Extremisation of the Lagrangian with respect to the undetermined functions which are contained in the chosen trial function yields a set of ordinary differential equations which have to

---

1 Some arguments in favour of such an assumption are discussed by Dr. Schiff (7) and in Chapter IV.

2 This method is described in Chapter V.

Table VI-A. Variational Approximations to Lowest Odd-Parity Axially Symmetric State.

Comparison function	Optimal values of parameters						Mass
	A	α	β	μ	σ	b	
1. $A\,z\,e^{-\alpha r}$	$\frac{32}{\sqrt{3}}$	$\sqrt{3}$					5.47
2. $A\,z\,e^{-\alpha r^{\beta}}$	3.03	2.76	1.8				4.72
3. $A\,z'\,e^{-r'}$	$\frac{32}{3}$	$\sqrt{5}$	$\sqrt{\frac{5}{3}}$				4.41
4. $A\,f_s$	.833						4.14
5. $A\left[P_1(\mu')r'e^{-r'} + P_3(\mu')r'^3e^{-\mu r'} + b\,P_3(\mu')r'^5e^{-\sigma r'}\right]$	6.48	2.08	2.18	1.8844	2.807	.6939	3.96

In the table:

$\vec{r}\,' \equiv (x', y', z') \equiv (\alpha x, \alpha y, \beta z)$, $\quad r' \equiv |\vec{r}\,'|$, $\quad \mu' \equiv \frac{z'}{r'}$.

P_n : Legendre polynomial of degree n.

Mass is in units of $\frac{mc^3\hbar}{g^2}$.

The function f_s was obtained by Strickfadden (9) as a solution of $\Delta\phi - \phi + \phi_T^{\,3} = 0$ using Greens function methods. ϕ_T is the comparison function no. 1 of the table. The results for comparison function no. 5 (previously unreported) are due to Dr. Schiff.

be solved numerically. The procedure does have the advantage that the set of trial functions so obtained satisfy identically some of the integral relations which are satisfied by the exact solutions of $\Delta\phi = \phi - \phi^3$.[1] In particular they satisfy the "orthogonality" relation (VI-1), suggesting that the various "excited" trial eigensolutions actually furnish approximations to corresponding "excited" nonspherical states of the field equation, if such exist.

The following types of trial functions were considered:

$$\phi_A = R(r)\ P_1(\mu)$$
$$\phi_B = r^m\ e^{-\alpha r}\ Y(\mu)$$
$$\phi_C = R(r)\ P_1(\mu) + Q(r)\ P_3(\mu)$$
$$\phi_D = (r^2 + b\ r^3)e^{-\alpha r}\ Y(\mu)$$
$$\phi_E = R(r)\ P_1(\mu) + Q(r)P_3(\mu) + S(r)P_5(\mu)$$

where $P_n(\mu)$ are Legendre polynomials, α, b and m[2] - parameters and R, Q, S and Y functions for variation. Details of the calculations together with graphs of the (numerically obtained)

1 This is shown for each of the trial functions considered in Appendices K, L, M and Ø.

2 m takes on integral values only.

optimal forms of the functions R, Q, S and Y are given in Appendices K, L, M, N and O. The results for the first few odd-parity eigenstates are listed in Table VI-B.

In the table A_n, B_n, C_n and $Y^{(n)}(\mu=1)$ are in units of $\frac{\hbar c}{g}$, while M is in units of $\frac{mc^3\hbar}{g^2}$. The integer n is used to label the eigensolution in order of increasing M. (Except in the case of $\phi_C^{(n)}$ for $n > 2$, n-1 is equal to the number of nodes of the functions $R^{(n)}$, $Q^{(n)}$, $S^{(n)}$, and $Y^{(n)}$.)

The nodeless optimised variational trial functions presented in table VI-B seem to approximate the lowest two-particle state[1] rather than a possible single particle state of odd parity, whose existence is suggested by the results presented in table VI-A (especially the trial function 5).

---

1 vis. $\phi_{12} = \phi_0(|\vec{r} + \vec{R}|) - \phi_0(|\vec{r} - \vec{R}|)$, where ϕ_0 is the lowest spherically symmetric one-particle state (p. 52) and $|\vec{R}| \to \infty$. Recall that $M(\phi_0) = 1.503 \frac{mc^3\hbar}{g^2}$ hence $M(\phi_{12}) = 3.006 \frac{mc^3\hbar}{g^2}$ (Reference (10)).

Table VI-B. Variational Approximations to Odd Parity Neutral Particlelike Solutions.

A. $\phi_A^{(n)} = R^{(n)}(r)P_1(\mu)$ $[R^{(n)}(r) \to A_n k_1(r)$ as $r \to \infty]$

n	A_n	M
1	± 10.08	4.87
2	± 73.64	16.60
3	±394.82	37.88

B. $\phi_B^{(n)} = r^m e^{-\alpha r} Y^{(n)}(\mu)$

n	$Y^{(n)}(\mu=1)$	m	α	M
1	± 36.66	2	2.19	3.76
2	± 19.59	2	1.32	31.33
3	± 10.70	2	0.874	109.8

C. $\phi_C^{(n)} = R^{(n)}(r)P_1(\mu)+Q^{(n)}(r)P_3(\mu)$ $[R^{(n)}(r) \to A_n k_1(r)$, $Q^{(n)}(r) \to B_n k_3(r)$ as $r \to \infty]$

n	A_n	B_n	M
1	± 10.15	± .8618	3.63
2	± 80.00	± 25.75	10.72
3	± 70.46	∓ 27.34	26.75
4	±276.15	± 63.46	38.92

D. $\phi_D^{(n)} = (r^2 + br^3)\, e^{-\alpha r}\, Y^{(n)}(\mu)$

n	$Y^{(n)}(\mu=1)$	α	b	M
1	±30.47	1.76	-0.228	3.73

E. $\phi_E^{(n)} = R^{(n)}(r)P_1(\mu) + Q^{(n)}(r)P_3(\mu) + S^{(n)}(r)P_5(\mu)$

$[R^{(n)} \to A_n k_1(r)$, $Q^{(n)} \to B_n k_3(r)$ and $S^{(n)} \to C_n k_5(r)$ as $r \to \infty$ $]$

n	A_n	B_n	C_n	M
1	± 11.30	± 1.4137	±.048532	3.31

Attempts were made to obtain odd-parity eigensolutions of (III-4) using the numerical technique described in Chapter V. The difference equation (V-13) corresponding to (III-4) was solved by iteration assuming that the eigensolutions have the asymptotic forms

a) $\phi_1 = A\, k_1(r) P_1(\mu)$ [1]

and

b) $\phi_2 = A_1 k_1(r) P_1(\mu) + A_3 k_3(r) P_3(\mu)$.

For case a) this was done using A = .1, .2, ..., 50. No meaningful results were obtained. It was found that as $r \to 0$, the "solutions" oscillate violently for all values of A considered; the amplitude of these "oscillations" increases with decreasing r.[2]

A number of reasons for such behaviour come to mind:

1. No well behaved, global, odd-parity solutions of equation (III-4) exist, which are asymptotic to zero. The results of variational calculations, however, suggest the contrary.[3]

---

1 P_n are Legendre Polynomials while k_n are spherical Hankel functions with imaginary argument (see p. 44).

2 A sample result, for A=10 is listed in Appendix P, together with plots of $\phi(r,\mu)$ for some (fixed) values of r.

3 See also the arguments given in Chapter IV, and (7).

2. The well behaved solutions are "obscured", in the numerical solution, by the possible existence of solutions which are unstable and/or divergent near the origin. Since no such solutions exist in the case of spherical symmetry, it would be curious indeed if such appeared in the odd-parity case.

3. No solutions exist with the asymptotic form ϕ_1[1].

4. The iterative method of solving the finite difference approximation to (III-4) which is used here, is not an efficient and accurate method for this type of equation. Some comments on the method are given in Chapter V. As noted there, the method has been used successfully for certain linear odd parity eigenvalue problems.

If the third possibility is taken to be the most probable, then a more realistic asymptotic form that ϕ_1 is necessary. This may be obtained from the variational comparison functions 4 of table VI-A and ϕ_C of table VI-B, which are of the form ϕ_2 indicated on p. 58 . In each case only a small region of the A_1, A_3 plane was investigated[2] in

---

1. It will be recalled that the general asymptotic form of the solutions is
$$\phi = \sum_{\ell=0}^{\infty} A_\ell k_\ell(r) P_\ell(\mu).$$

2. It is clear that the amount of machine time necessary to investigate exhaustively even the case of two asymptotic parameters A_ℓ (say A_1 and A_3) is too large to merit such an undertaking.

the vicinity of those values for which ϕ_2 corresponds to the indicated trial functions. The results obtained were in all cases qualitatively similar to those obtained with the asymptotic form ϕ_1 . Finally, the variational trial function 5 of table VI-A was used to approximate an odd parity solution in the asymptotic region. In this case also the numerical results obtained could not be interpreted as approximating a well behaved solution.

As noted in Chapter III, neutral particlelike solutions are possible when space is subdivided into alternating regions of linearity, where

$$\Delta\phi = 0$$

and

$$|\vec{A}|^2 = \phi^2 - 1 \text{ ,}$$ [1]

and nonlinearity, where

$$\Delta\phi = \phi - \phi^3$$

and

$$\vec{A} = 0.$$

The linear regions are bounded by concentric spheres[2] of radii R_n and R_{n+1} $(0 < R_n < R_{n+1} < \infty)$, with $\phi = \pm 1$ on R_n

1. Note that $|\vec{A}|$ is imaginary since $|\phi| < 1$ in the linear region.

2. We consider here only spherically symmetric solutions.

and $\phi = \mp 1$ on R_{n+1} [1], so that every linear region must contain a zero (node) of the solution. Thus a solution possessing 3 nodes, say, can be one with 0, 1, 2 or 3 regions of linearity. There are, then, a total of eight possible distinct neutral particlelike solutions having 3 nodes. In general, if (N) denotes the class of solutions having N zeros, then the greatest number of distinct neutral particlelike solutions which can belong to this class is

$$1 + \sum_{k=1}^{N} \frac{N(N-1)\ \ldots\ (N-k)}{k!} .$$

The mass corresponding to a given particlelike solution is, from (III-3)[2]

$$M = \sum_{n\ell} \frac{1}{4\pi} \int_{V_{n\ell}} [\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\phi^2 - \tfrac{1}{4}\phi^4]\ d^3x + \sum_{\ell} \frac{1}{4\pi} \int_{V_\ell} \tfrac{1}{2}(\nabla\phi)^2 d^3x,$$

where $\sum_{\ell}$ and $\sum_{n\ell}$ denote summation over all linear regions V_ℓ, and all nonlinear regions $V_{n\ell}$ respectively.

---

1 In addition $\frac{d\phi}{dr}$ must be continuous on R_n and R_{n+1}.

2 Recall that N=-1, K=0 and M=-L for neutral particlelike solutions.

This expression can be reduced, with the aid of the integral relation (VIII-2), to

$$M = \sum_{n\ell} \frac{1}{16\pi} \int_{V_{n\ell}} \phi^4 \, d^3x ,$$

implying that the linear regions contribute nothing to the rest mass of the particle.

In the table below we list the (numerically obtained) results for some compound neutral particlelike solutions:[1]

Type	A	$\phi(0)$	M
(1;L)	± 17.01	∓ 14.15	9.46
(2;L,NL)	± 83.85	± 29.15	28.71
(2;NL,L)	± 80.90	± 29.42	28.58
(2;L,L)	± 80.89	± 29.42	28.58
(3;L,NL,NL)	±375.0	∓ 49.43	63.97
(3;NL,L,NL)	±374.3	∓ 49.47	63.96
(3;L,L,NL)	±374.3	∓ 49.47	63.96
(3;NL,NL,L)	±342.2	∓ 50.18	63.44
(3;L,NL,L)	±342.2	∓ 50.18	63.44
(3; NL,L,L)	±341.7	∓ 50.21	63.43
(3;L,L,L)	±341.7	∓ 50.21	63.43

1 In the table A, $\phi(0)$ and M are in units of $\frac{\hbar c}{g}$, $\frac{mc^2}{g}$ and $\frac{mc^3 \hbar}{g^2}$ respectively. (2;NL,L) refers to a solution with 2 zeros, with the region around the first (where $|\phi| < 1$) being nonlinear and around the second - a linear one.

These results differ very little from those for the corresponding solutions listed on page 52.[1] This is because the regions of linearity (at least for the first few solutions considered here) are near the origin and/or small in extension and $|\phi| < 1$ there. In fact the only perceptible difference (to 3 significant figures) in corresponding mass values occurs where there is a linear region at some distance from the origin[2] (for example (2;NL,L) or (3;NL,NL,L)). A sample curve, (2;NL,L), showing the linear and nonlinear regions, is plotted in figure 6.5.

---

1 For this reason they are not plotted, since on a scale like that used in figures 6.1 to 6.4 they are indistinguishable.

2 The region near the origin contributes little to the mass because of the factor r^2 in the integrand of

$$M = \frac{1}{4} \int \phi^4 r^2 dr \ .$$

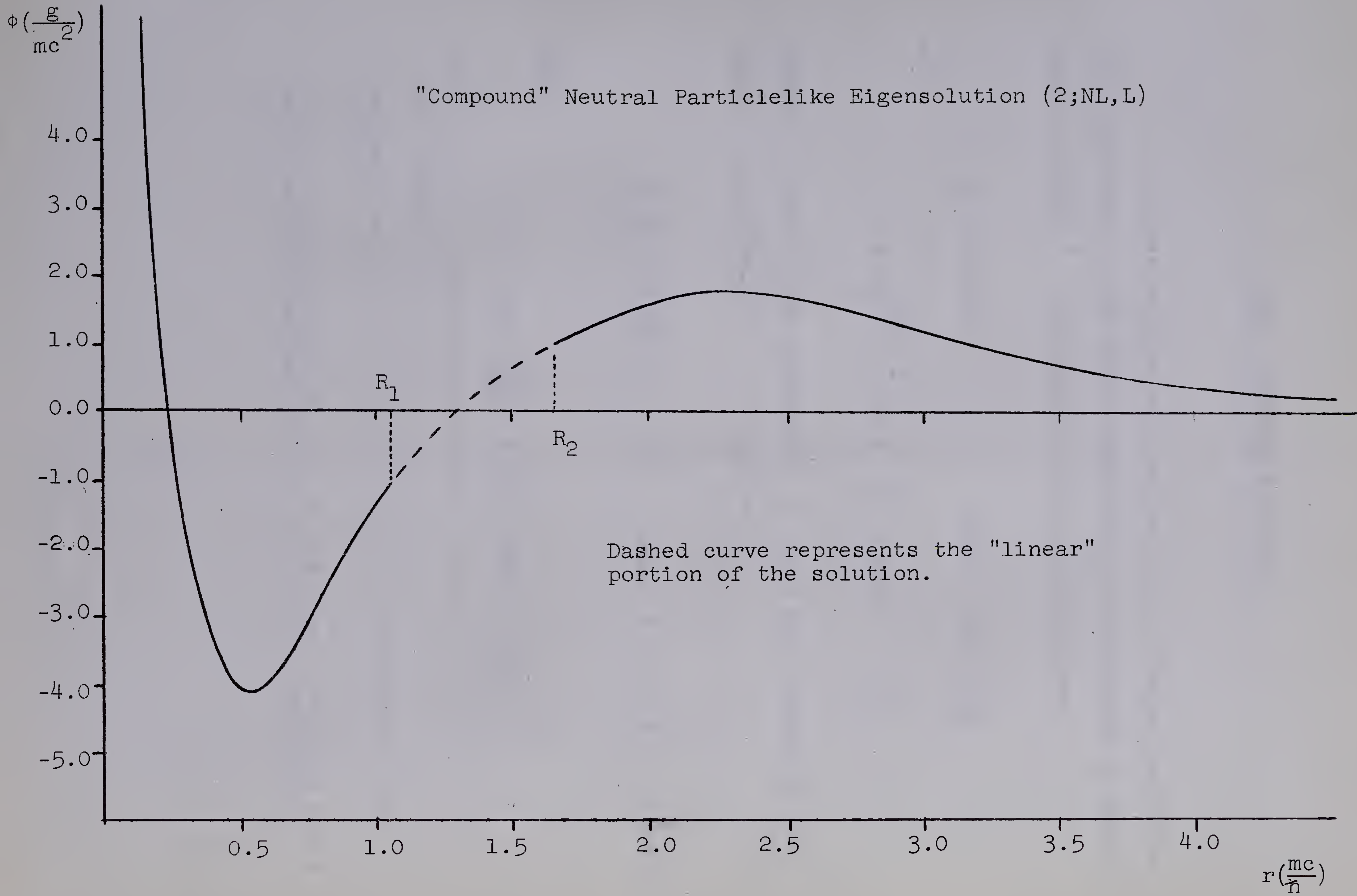
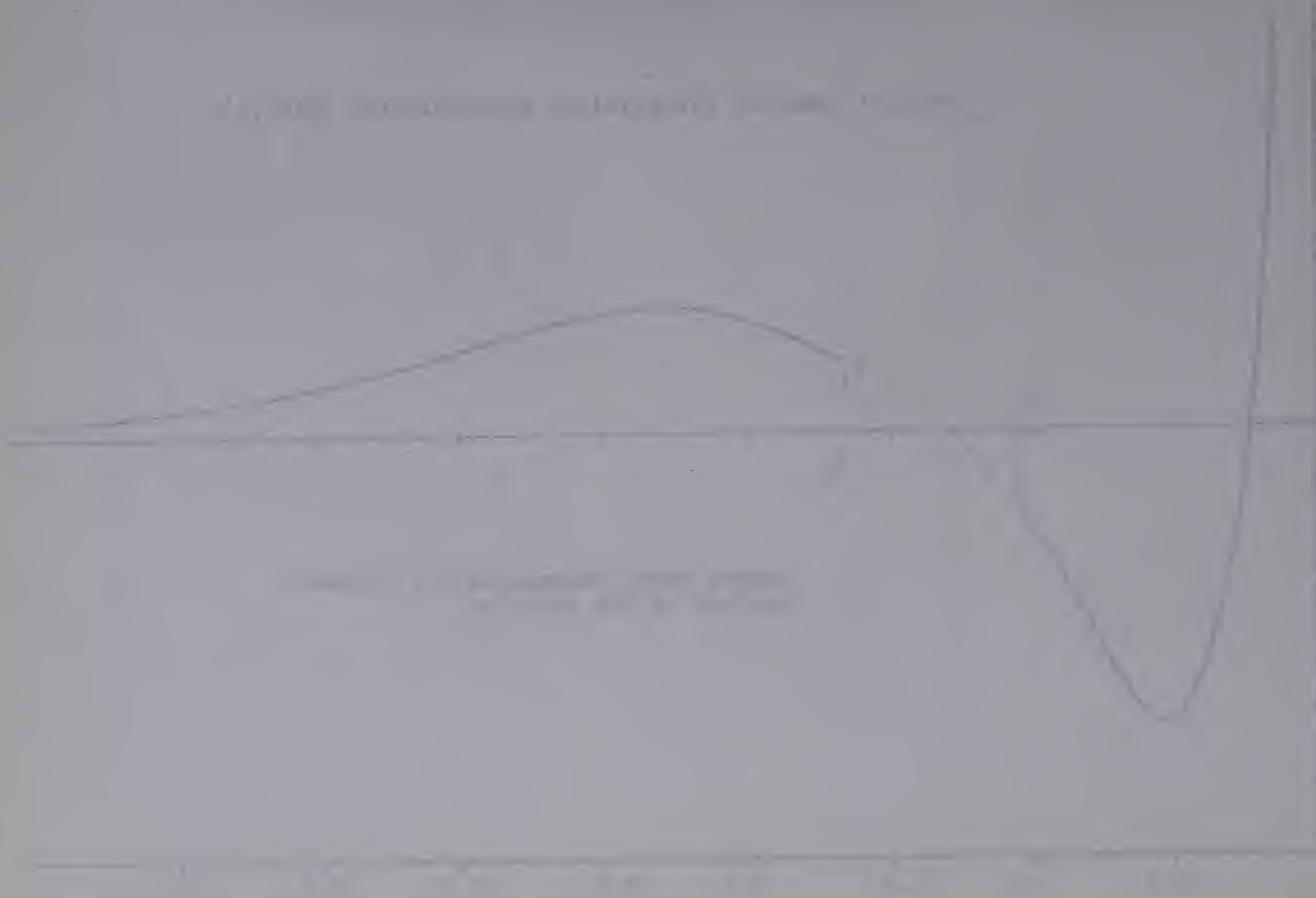

"Compound" Neutral Particlelike Eigensolution (2;NL,L)
$\phi(\frac{g}{mc^2})$
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-5.0
R_1
R_2
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$r(\frac{mc}{\hbar})$
Dashed curve represents the "linear" portion of the solution.

Fig. 6.5



VII. CHARGED PARTICLES

In this section we consider static charged particle-like solutions of the field equations in the absence of the magnetic field.[1] The simplest such solutions are

$$A_i = (\vec{0},\ i\phi) \qquad \text{when} \quad 0 \le r \le \frac{b}{\alpha}$$

and

$$= (\pm\sqrt{\phi^2 - 1}\ \hat{r},\ i(a + \frac{b}{r})\ ) \ \text{when} \quad r \ge \frac{b}{\alpha}\ ,$$

where a and b are constants[2], $\hat{r}$ is a unit vector in the radial direction and ϕ a solution of:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = \phi - \phi^3 \qquad \text{when} \quad 0 \le r \le \frac{b}{\alpha}\ , \qquad \text{(VII-1)}$$

with

$$\phi = -1 \quad \text{and} \quad \frac{d\phi}{dr} = -\frac{\alpha^2}{b} \quad \text{when} \quad r = \frac{b}{\alpha}\ .$$[3]

---

1 See Chapter III, p. 21.

2 We shall use both α and $a = \pm 1 - \alpha$ throughout this chapter.

3 Since for every solution ϕ, $-\phi$ is also a solution it is sufficient to consider $\phi = -1$ on $r = b/\alpha$ instead of $\phi = \pm 1$.

The Lagrangian associated with each solution is, from (III-3),

$$L = \int_0^{b/\alpha} [\tfrac{1}{2} (\tfrac{d\phi}{dr})^2 + \tfrac{1}{2} \phi^2 - \tfrac{1}{4} \phi^4 - \tfrac{1}{4}]r^2 dr + \tfrac{1}{2} \alpha b. \qquad \text{(VII-2)}$$

The constant N, which appears in (III-3), is taken to be unity so that the energy-momentum tensor, in the region where $1 + A_i{}^2 = 0$, will be the same as in the Maxwell theory. K is taken to be $\frac{1}{4}$ so that the Lagrangian and hence the energy, which is linearly related to it[1], is finite. The corresponding mass is then

$$M = -L - ab.$$

The spin and orbital angular momentum[2] are zero as for neutral particles.

As shown in Chapter IV, there is a solution of (VII-1) for each value of $\phi(0)$ (the constants α and b, of course, change with $\phi(0)$).[3] These solutions may be grouped

---

1 Equation (I-12).

2 These were defined in Chapter I.

3 $\phi(0)$ is the amplitude of ϕ at the origin.

together according to the number of nodes they possess on the lines $\phi = 0$ and $\phi = -1$. We shall use the notation (M, N) to denote the group of solutions of (VII-1) having M nodes on the line $\phi = 0$ and N nodes on the line $\phi = -1$.[1]

There exists, then, a continuum of well behaved solutions of (VII-1) within each group (M, N). They may be characterised by a single scalar parameter such as $\phi(0)$ or b (the charge parameter).[2]

A discrete set of particlelike solutions can be determined, for example, by imposing the requirement that the charge parameters be all the same (in absolute value) or have small integral ratios. It is preferable, however, to use a general principle to determine a discrete set of particlelike solutions, rather than an artificially imposed one like that cited above. For this purpose the principle of least action may be invoked: Those solutions for which L is a minimum with respect to the parameter (or parameters) characterising the continuum (that is, the most stable solutions) will be considered as representing charged particles.

---

1 Some such solutions are plotted in figures 7.2 - 7.5.

2 The parameter a(or α) is not convenient for characterising the solutions, as is shown below.

Thus, the discrete set of spherically symmetric charged particlelike solutions is characterised by the discrete set $\{b_i\}$, for which $L(b)$ is a local minimum, and the mass $M_i = -L(b_i) - b_i a(b_i)$ is positive.

If $\phi(b)$ and $\phi(b + \delta b)$ are two "neighbouring" solutions of (VII-1) (δb is assumed to be small), then the change in L corresponding to $\delta\phi = \phi(b + \delta b) - \phi(b)$ is, as pointed out in Chapter V, p.35,

$$\delta L = \frac{1}{4\pi} \oint_{S_\infty} \delta\phi \; \nabla\phi \cdot d\vec{s} = - b \, \delta a \, ,$$

since $\phi = a + \frac{b}{r}$ for $r > \frac{b}{\alpha}$. S_∞ is a spherical surface of radius $R \to \infty$. In the limit as $\delta b \to 0$ (hence $\delta\phi \to 0$) this becomes

$$\frac{dL}{db} = -b \frac{da}{db} \; .$$ [1]

Thus, for $b \neq 0$, L and a have concurrent extrema.

For the discrete set of charged particlelike solutions of (VII-1), the action principle

---

1 Equivalently $\dfrac{dL}{d\phi(0)} = -b \dfrac{da}{d\phi(0)}$.

$$\delta L + b\,\delta a = 0$$

then becomes $\delta L = 0$ subject to $\delta a = 0$.

This may be cast into a simpler form by the use of a Lagrange multiplier; that is, the above statement is equivalent to requiring that

$$\delta L + b'\delta a = 0 \qquad \text{(VII-3)}$$

where b' is an undetermined multiplier. Since for exact solutions of (VII-1) $b' = b$, it is clear that a variational calculation based on (VII-3) will yield approximations to the discrete set of charge particlelike solutions, and the multiplier b' will furnish an approximation to the corresponding "charge" b.[1]

L(b) exhibits a minimum for each group of solutions (M,N); an example of such a curve, corresponding to the group of solutions (1,1) is plotted in figure 7.1 together with a(b) and M(b) for this group.[2]

---

1 If no subsidiary conditions (on the comparison function) are imposed then (VII-3) will yield an approximation to the "ground state" charged particle, that is to that solution which corresponds to the smallest of $\{L(b_i)\}$.

2 The behaviour of these functions for other groups of solutions is qualitatively very similar (except for the group (0,1) for which $L_{min}=0$ at b=0 and a(b) exhibits no extremum). In general M(b) exhibits no extrema.

L(b), a (b), and M(b) for the group (1,1)

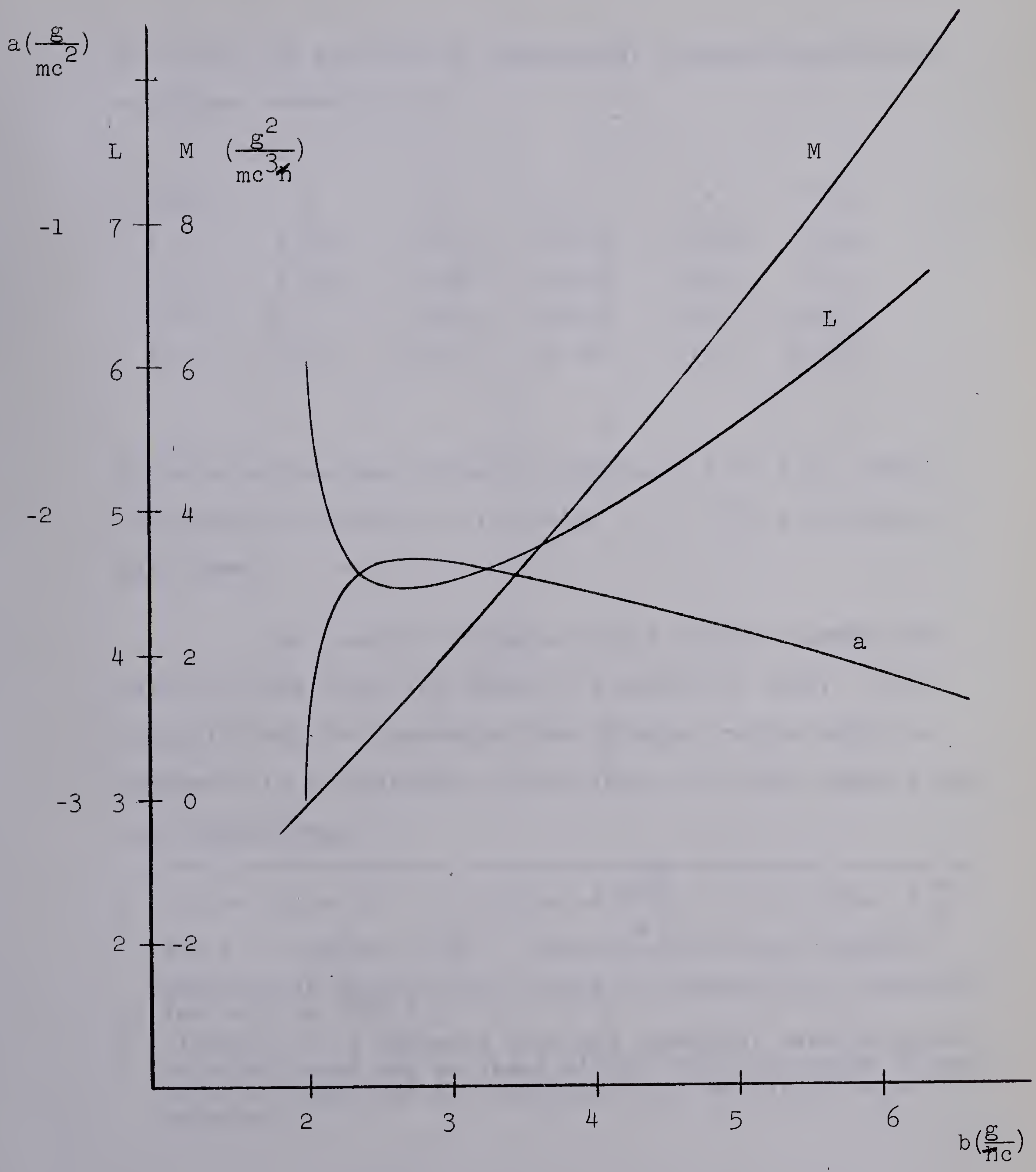


Fig. 7.1

The first few spherically symmetrical charged particlelike solutions correspond to:[1]

group	L	b	a	M	$\phi(0)$
(1,1)	4.48	2.76	-2.177	1.530	6.98
(1,3)	7.46	7.89	-2.168	9.65	7.00
(2,2)	21.1	8.02	-3.234	4.80	-18.61
(3,3)	57.15	15.70	-4.246	9.50	35.68

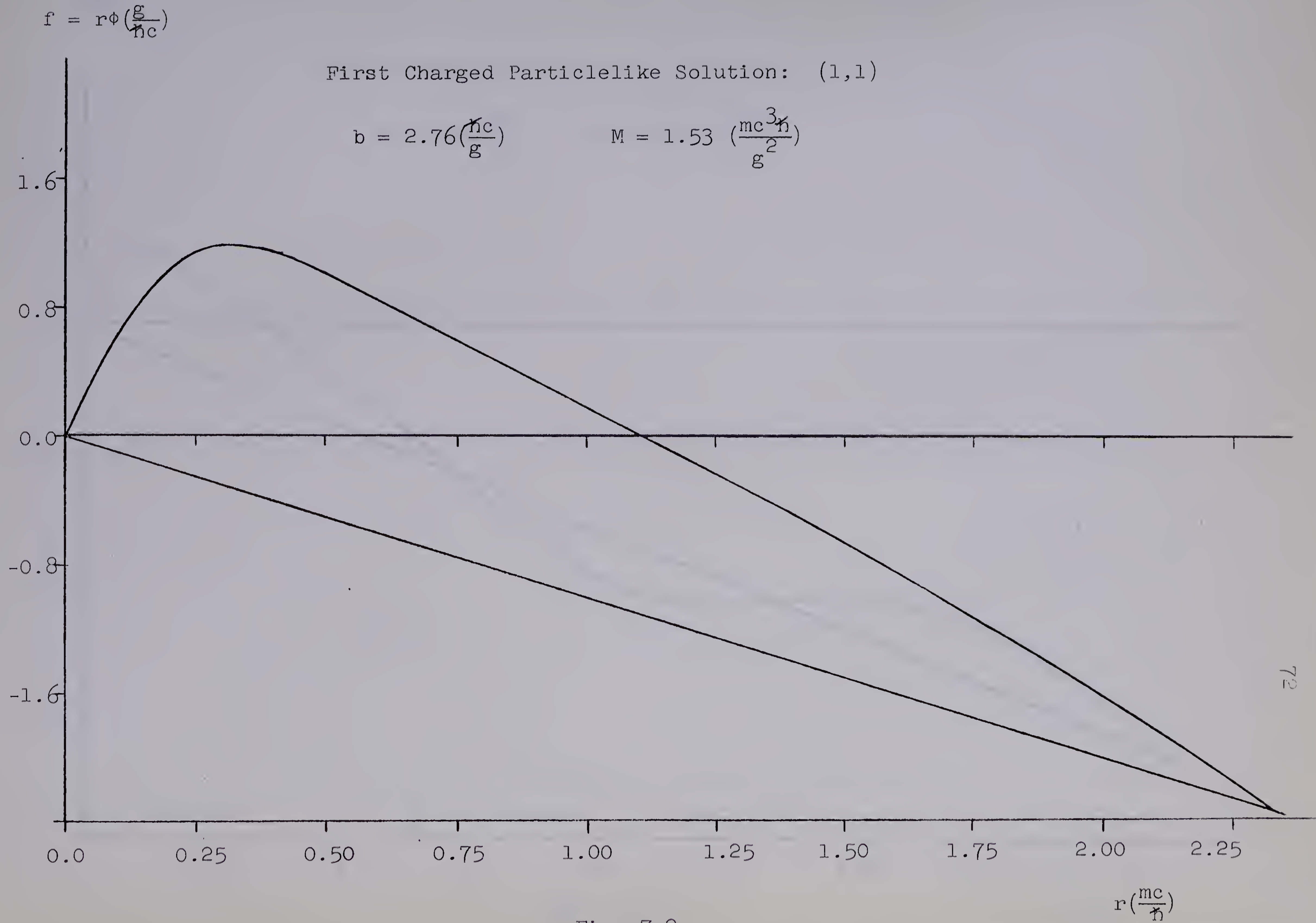
These solutions are plotted in figures 7.2 to 7.5. (Only that portion of solution for which $r \leq \frac{b}{\alpha}$ is plotted in each case).

The (nearly) integral ratios of the charges (at least for the first few states) is worthy of note. It is possible that the departure from integral ratios which is apparent, is a reflection of the fact that these results are only approximate.[2]

---

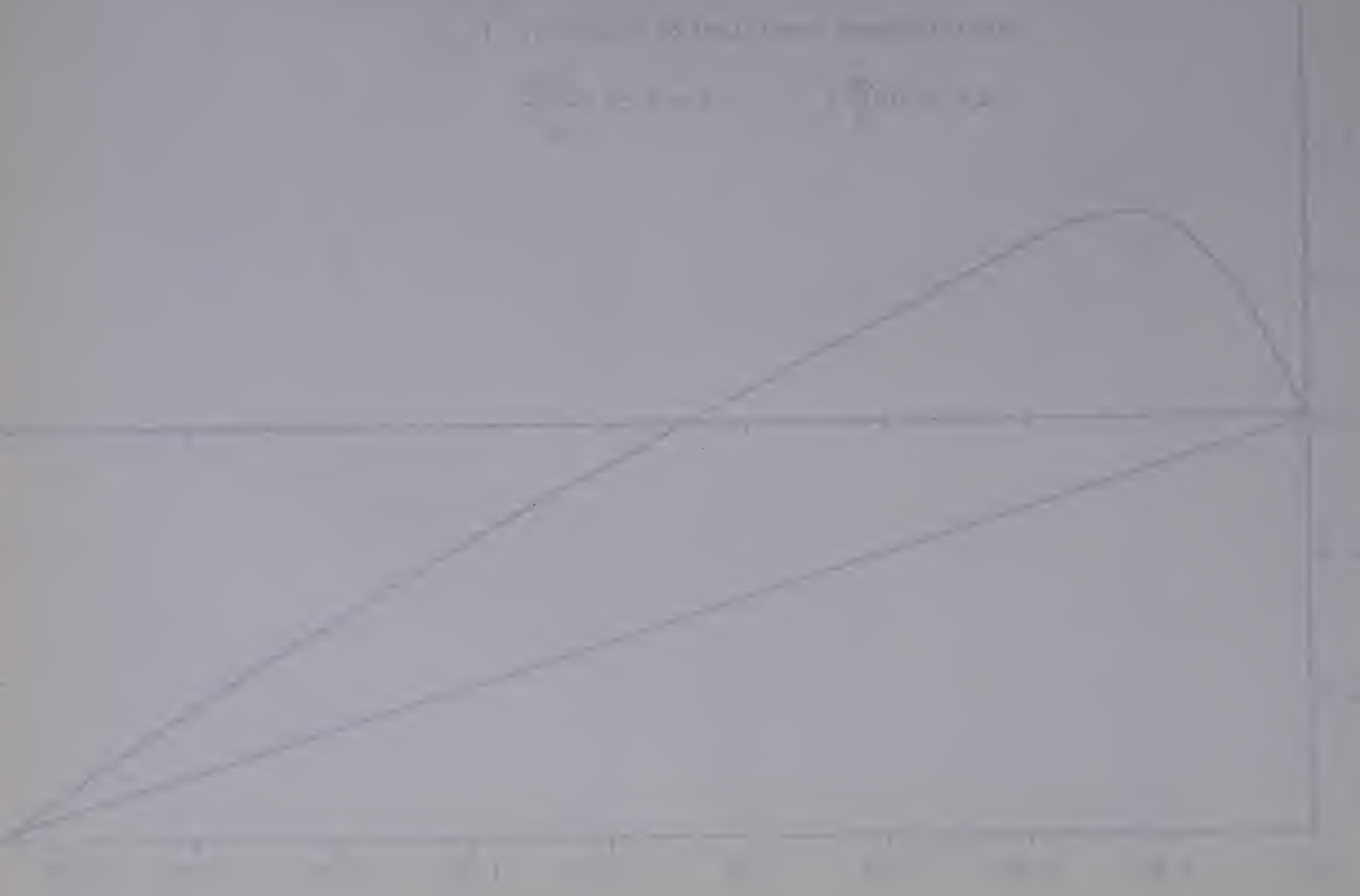
1 In the table M is in units of $\frac{mc^3\hbar}{g^2}$, b in units of $\frac{\hbar c}{g}$ and a in units of $\frac{mc^2}{g}$. These results were obtained numerically by Dr. Schiff using the Runge-Kutta technique for solving (VII-1).

2 Although it is believed that all numerical results given in this thesis are at least within 1% of the exact values, it is not possible to determine this "deviation" with certainty.



$f = r\phi(\frac{g}{\hbar c})$
First Charged Particlelike Solution: (1,1)
$b = 2.76(\frac{\hbar c}{g})$
$M = 1.53\ (\frac{mc^3\hbar}{g^2})$
1.6
0.8
0.0
-0.8
-1.6
0.0
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
2.25
$r(\frac{mc}{\hbar})$

Fig. 7.2



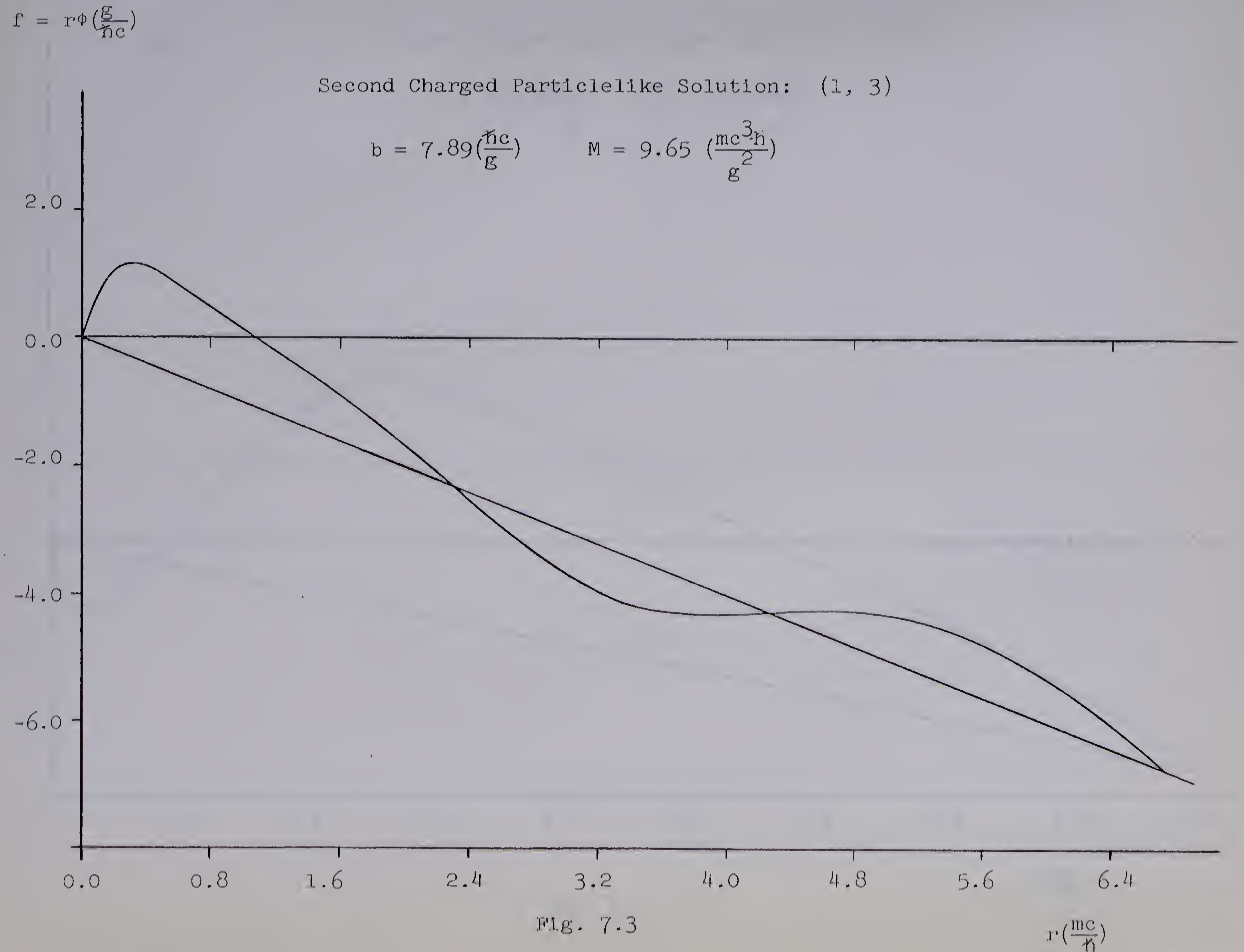
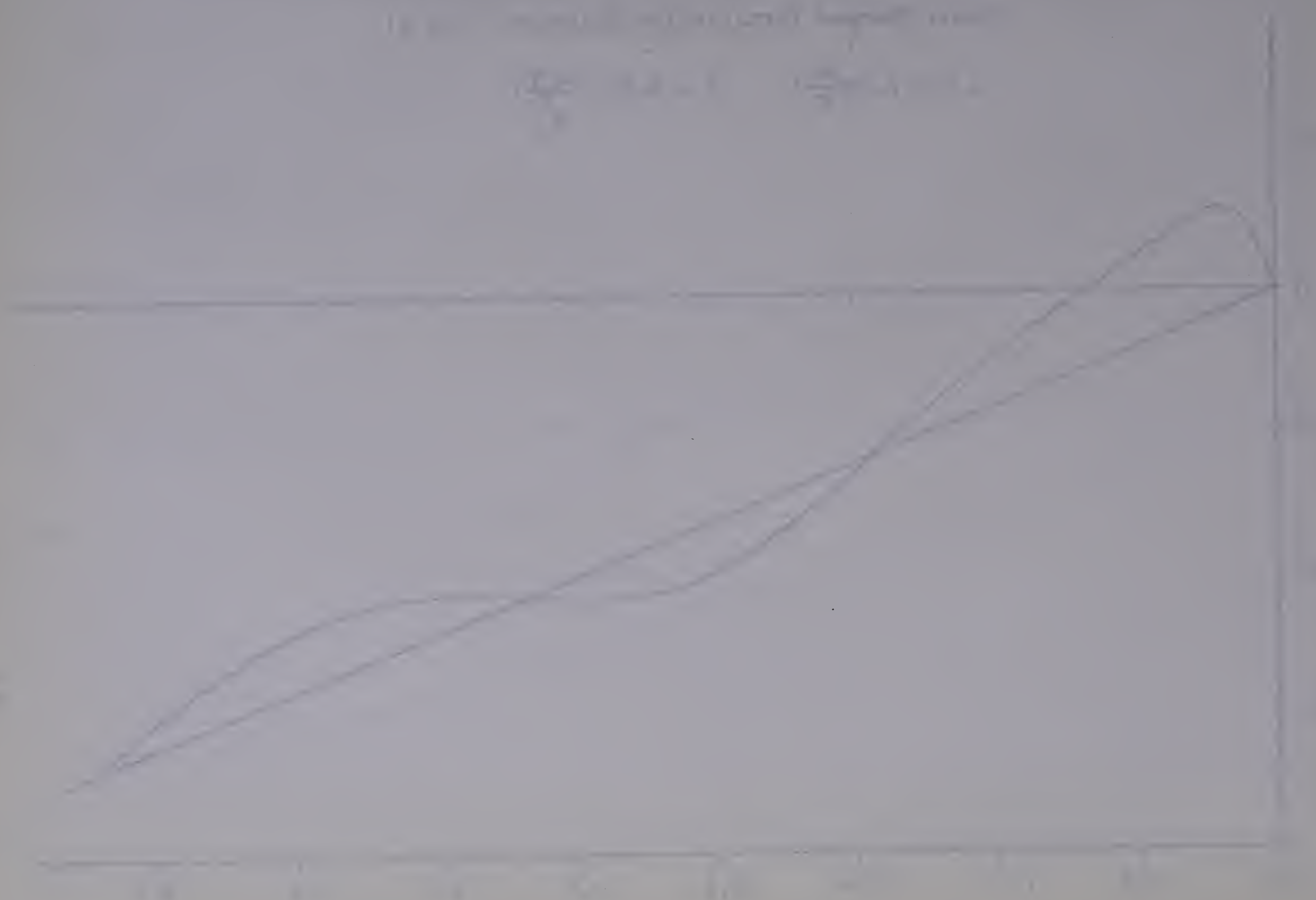
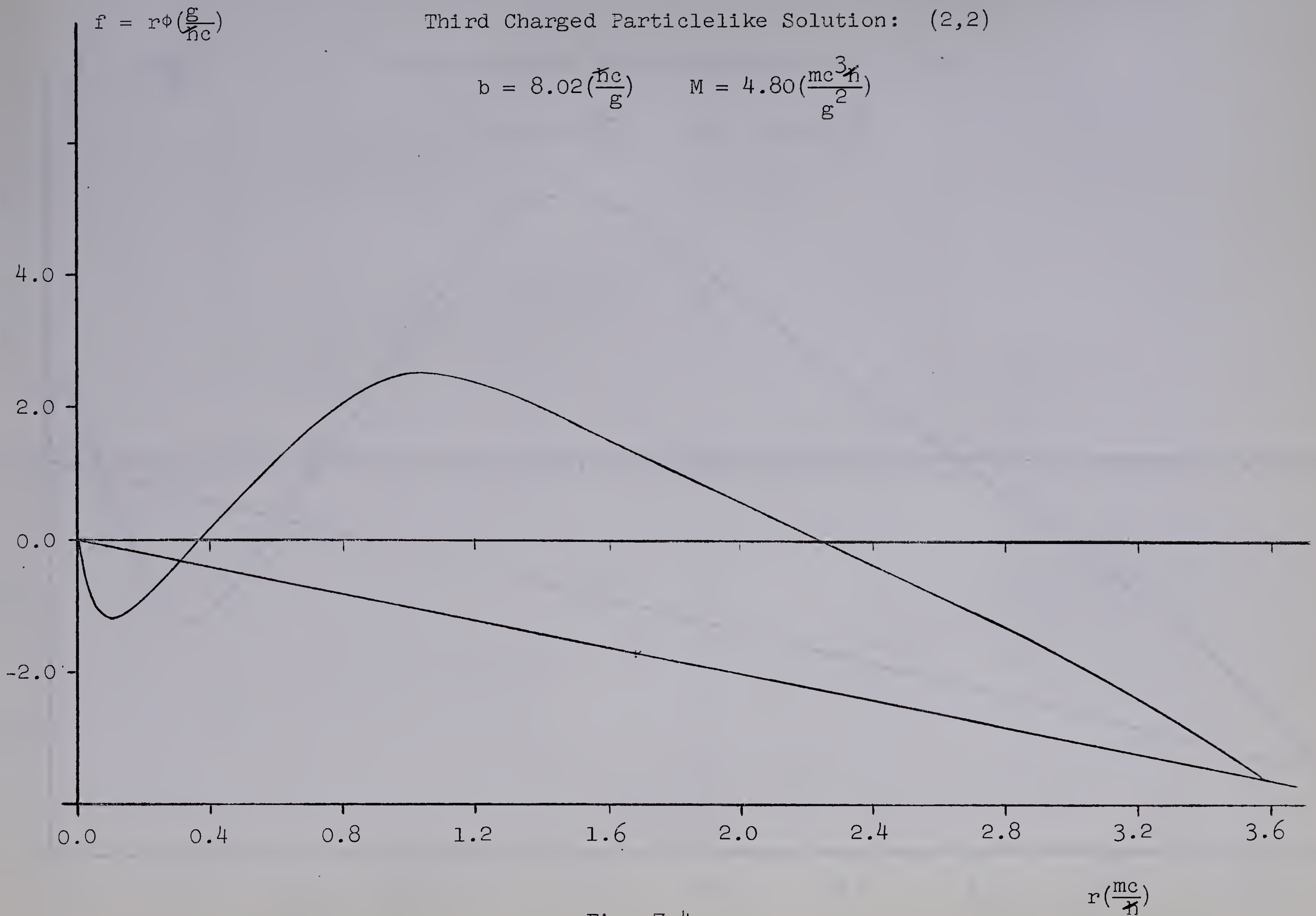

$f = r\phi(\frac{g}{\hbar c})$
Second Charged Particlelike Solution: (1, 3)
$b = 7.89(\frac{\hbar c}{g})$
$M = 9.65\ (\frac{mc^3\hbar}{g^2})$
2.0
0.0
-2.0
-4.0
-6.0
0.0
0.8
1.6
2.4
3.2
4.0
4.8
5.6
6.4
$r(\frac{mc}{\hbar})$

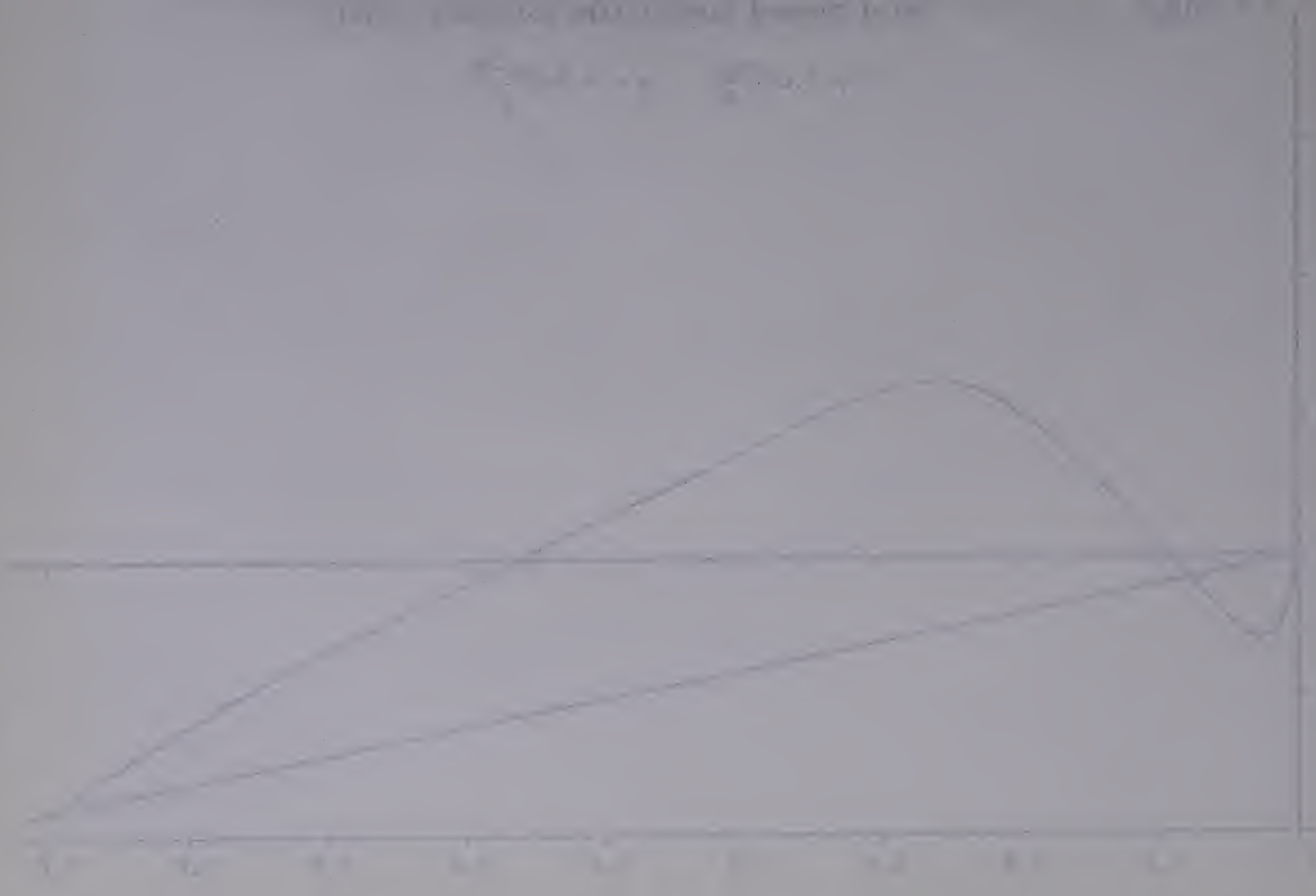
Fig. 7.3

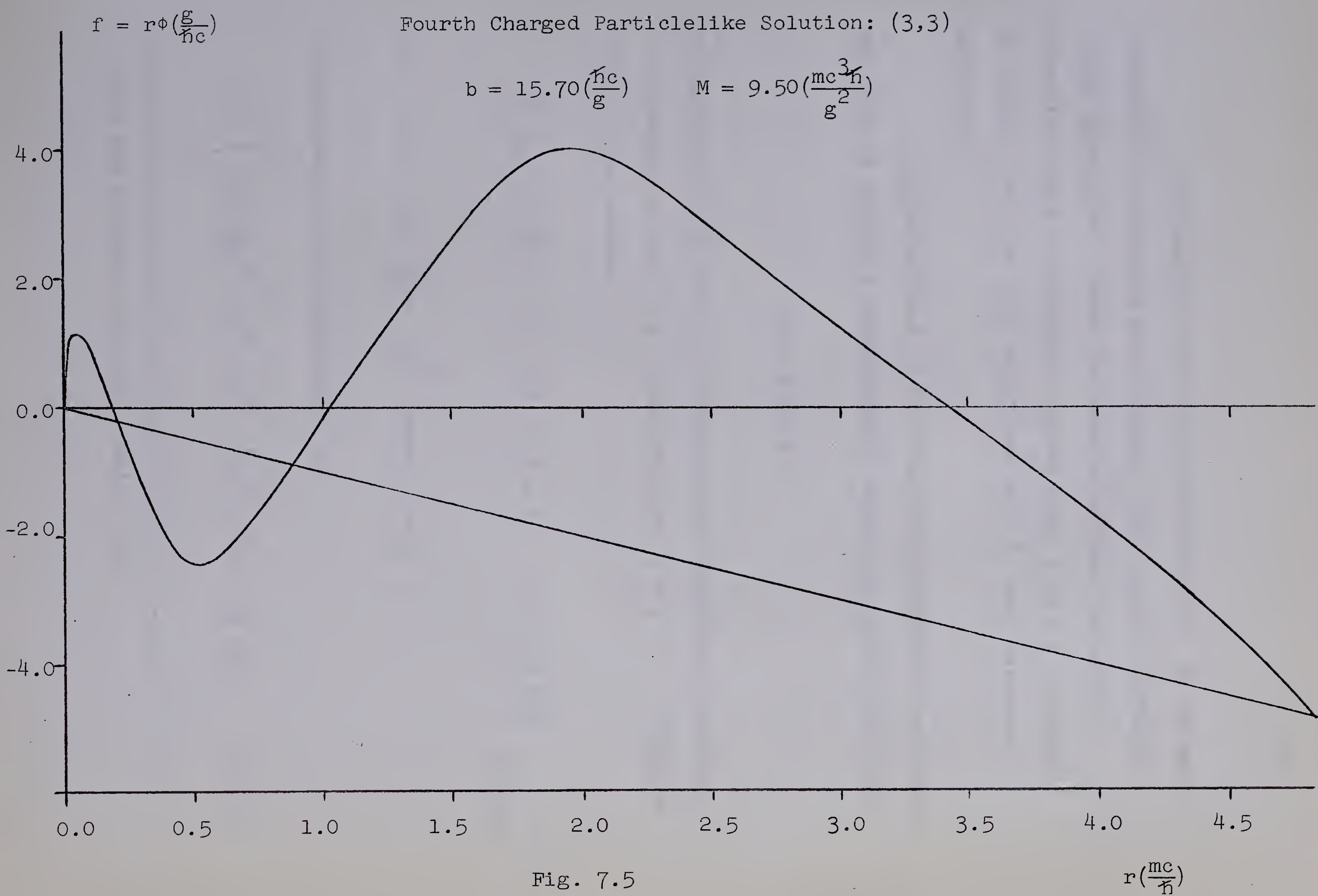




$f = r\phi(\frac{g}{\hbar c})$
Third Charged Particlelike Solution: (2,2)
$b = 8.02(\frac{\hbar c}{g})$
$M = 4.80(\frac{mc^3\hbar}{g^2})$
4.0
2.0
0.0
-2.0
0.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
$r(\frac{mc}{\hbar})$

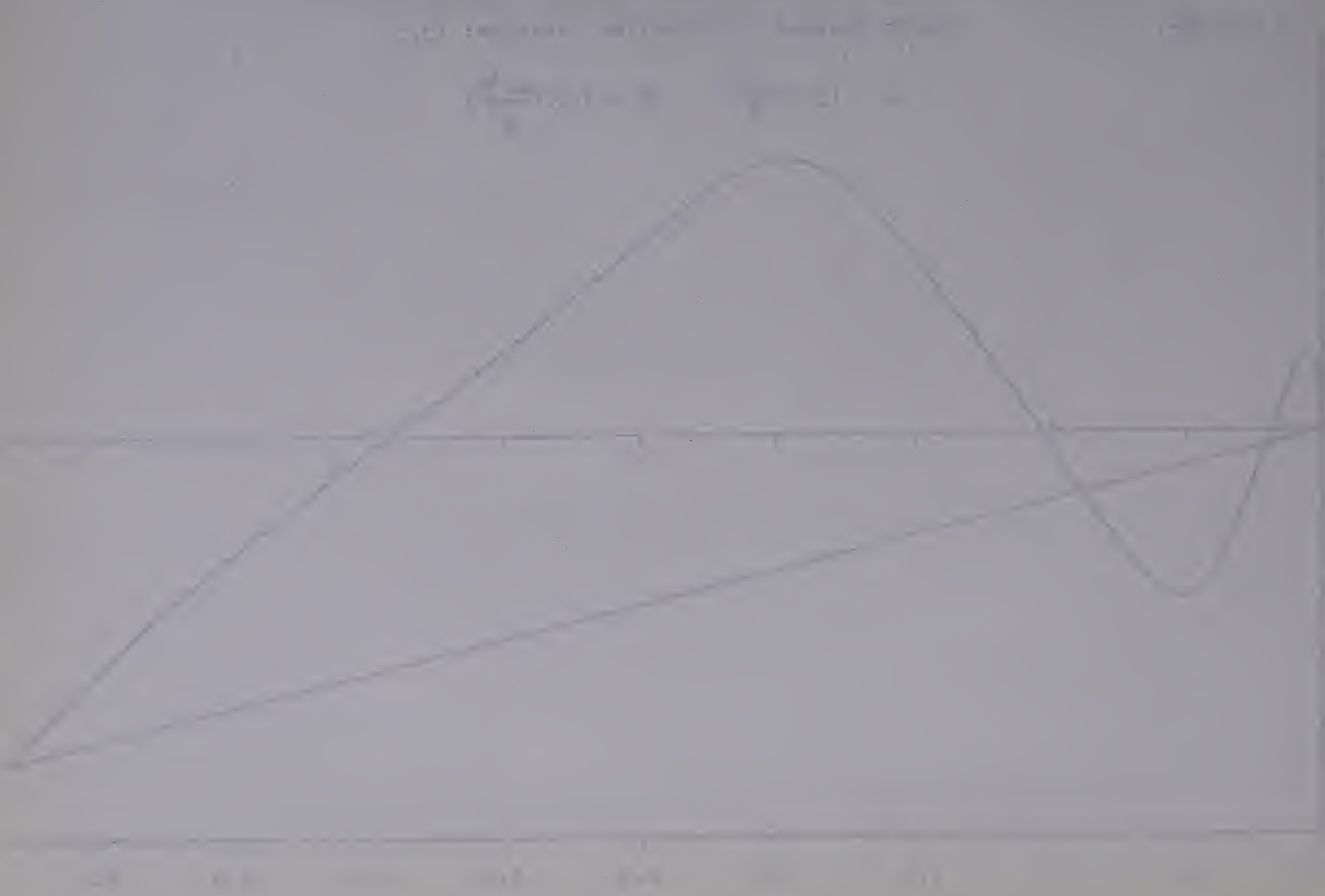
Fig. 7.4





$f = r\phi(\frac{g}{\hbar c})$
Fourth Charged Particlelike Solution: (3,3)
$b = 15.70(\frac{\hbar c}{g})$
$M = 9.50(\frac{mc^3\hbar}{g^2})$
4.0
2.0
0.0
-2.0
-4.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
$r(\frac{mc}{\hbar})$

Fig. 7.5



Solutions for which the difference $M - N$ is odd correspond to negative mass states, hence cannot represent actual particles. Since for such solutions the vector potential takes on imaginary values when $|\phi| < 1$ it might be conjectured that only solutions with $\vec{A}$ real represent particles.[1]

Approximations to the charged particlelike solutions may also be obtained variationally using the action principle

$$\delta L + b\,\delta a = 0.$$

To facilitate the choice of variational comparison functions it is useful to make the transformation $\phi = \chi - 1$. Equation (VII-1) then becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\chi}{dr}\right) + 2\chi - 3\chi^2 + \chi^3 = 0\ , \quad \text{if } 0 \leq r \leq \frac{b}{\alpha} \tag{VII-4}$$

with $\chi = 0$ and $\frac{d\chi}{dr} = -\frac{\alpha^2}{b}$ when $r = \frac{b}{\alpha}$.

Correspondingly, the Lagrangian is

$$L = \int_0^{b/\alpha} \left[\frac{1}{2}\left(\frac{d\chi}{dr}\right)^2 - \chi^2 + \chi^3 - \frac{1}{4}\chi^4\right] r^2 dr + \frac{1}{2}\alpha b\ . \tag{VII-5}$$

---

1 This was pointed out by Dr. Schiff (7).

Let $\chi = \lambda f(\mu r, \eta_i)$, λ, μ, η_i being amplitude, radial scale and other parameters for variation, and f an arbitrary integrable comparison function with a zero in the region $r > 0$.

From $\chi = 0$ at $r = \frac{b}{\alpha}$ we immediately get

$$\frac{\mu b}{\alpha} = c(\eta_i) \tag{VII-6}$$

where, for a given f, c is a function of η_i only, defined by $f(c, \eta_i) = 0$. Similarly, from $\frac{d\chi}{dr} = -\frac{\alpha^2}{b}$ when $r = \frac{b}{\alpha}$ we get

$$\lambda\mu \, v(\eta_i) = \frac{\alpha^2}{b} \tag{VII-7}$$

where the function $v(\eta_i) = -\left[\frac{\partial f(x, \eta_i)}{\partial x}\right]_{x = c(\eta_i)}$.

Because of the relations (VII-6) and (VII-7) all but (any) two of the parameters α, b, λ, μ, η_i are (initially) independent. We shall take α, μ, η_i to be the independent parameters, so that

$$L = L(\alpha, \mu, \eta_i) \quad ,$$

$$b(\alpha, \mu, \eta_i) = \frac{\alpha c(\eta_i)}{\mu}$$

and

$$\lambda(\alpha, \mu, \eta_i) = \frac{\alpha}{v(\eta_i) c(\eta_i)} \quad .$$

Then $\delta L - b\,\delta\alpha = (\frac{\partial L}{\partial \alpha} - b)\delta\alpha + \frac{\partial L}{\partial \mu}\,\delta\mu + \sum_i \frac{\partial L}{\partial \eta_i}\,\delta\eta_i = 0$

implies

$$\frac{\partial L}{\partial \alpha} = b \tag{VII-8}$$

$$\frac{\partial L}{\partial \mu} = 0 \tag{VII-9}$$

$$\frac{\partial L}{\partial \eta_i} = 0\ . \tag{VII-10}$$

The Lagrangian (VII-5) now simplifies to

$$L(\alpha,\mu,\eta_i) = \frac{\lambda^2}{2\mu}\,G_o(\eta_i) - \frac{1}{\mu^3}\,[\tfrac{1}{4}\lambda^4 G_4(\eta_i) - \lambda^3 G_3(\eta_i) + \lambda^2 G_2(\eta_i)]$$

where

$$G_o(\eta_i) = c^3 v^2 + G(\eta_i)$$

$$G(\eta_i) = \int_0^c (\frac{\partial f}{\partial x})^2 x^2\,dx$$

and

$$G_k(\eta_i) = \int_0^c [f(x,\eta_i)]^k\,x^2\,dx \quad \text{for } k=2,3 \text{ and } 4.$$

From (VII-8)

$$G_o\mu^2 = 6[\tfrac{1}{4}\lambda^2 G_4 - \lambda G_3 + G_2] \qquad \text{for } \lambda \neq 0, \tag{VII-11}$$

and from (VII-9)

$$G\mu^2 = \lambda^2 G_4 - 3\lambda G_3 + 2G_2 \tag{VII-12}$$

These relations may be solved for λ and μ as functions of η_1. Implicitly:

$$\mu = \left[\frac{6}{G_o}\left(\frac{1}{4}\lambda^2 G_4 - \lambda G_3 + G_2\right)\right]^{1/2}$$

where

$$(2c^3v^2-G)G_4\lambda^2 - 6(c^3v^2-G)G_3\lambda + 4(c^3v^2-2G)G_2 = 0 \ .$$

The corresponding expression for the Lagrangian is

$$L(\eta_1) = \frac{\lambda^2 G_o}{3\mu} \quad .$$

Results obtained in this way, for various choices of f, are tabulated below:[1]

$f(x,\eta_1)$	η_1	η_2	L	b	M	a
Numerical integration of (VII-1)			4.48	2.76	1.53	-2.177
$1-x^{\eta_1}$			no extremum found			
$1-2x+\eta_1 x^2$	.99463		4.76	1.90	.491	-1.857
$1-2x^{\eta_1}+\eta_2 x^{2\eta_1}$	.635	.99017	4.59	2.27	.930	-2.013
$\frac{1}{x^2+1} - \eta_1$	.0830		4.245	2.25	1.10	-1.95
$\cos(x+\eta_1)+\eta_2$	2.925	.999878	4.77	1.90	.556	-1.841

---

1 The extremisation of L with respect to η_1 was done numerically in all cases. Expressions for the integrals G_o, G_k for each function are listed in Appendix Q.

The variational procedure is not very efficient for obtaining approximations to the charged particlelike solutions. The choice of suitable comparison functions is restricted by the following:

(1) $f(x,\eta_i)$ must possess a zero for $x > 0$,

(2) c must be explicitly solvable in terms of η_i,

(3) the integrals G_o, G_k must be expressible in terms of elementary functions.

The computations are rather involved, and while the results for L are not unreasonable, the approximate values of the physically meaningful quantities M and b are in poor agreement with those obtained from numerical solution of VII-1.[1] It is clear that the variational procedure would be much more cumbersome for obtaining approximations to any non-spherical charged particlelike solutions.

We consider, finally, "compound" spherically symmetric charged particlelike solutions, that is solutions

---

1 This is, of course, because it is L that is extremised in the variational procedure not M or b. Thus, if the variational comparison function $\psi = \phi + \epsilon$(ϕ being the exact solution) then $|L(\psi) - L(\phi)|$ is of the order of ϵ^2 whereas for M or b the corresponding deviation is of the order of ϵ. This differs from the neutral particle case where $L = -M$.

where the regions in the vicinity of their zeros (and where $|\phi| < 1$) are linear regions such that

$$\Delta\phi = 0$$

and

$$|\vec{A}|^2 = \phi^2 - 1 < 0 \quad \text{there.}$$

The Lagrangian in this case becomes

$$L = \sum_{n\ell} \frac{1}{4\pi} \int_{V_{n\ell}} [\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\phi^2 - \tfrac{1}{4}\phi^4 - \tfrac{1}{4}]\, d^3x + \sum_{\ell} \frac{1}{4\pi} \int_{V_\ell} \tfrac{1}{2}(\nabla\phi)^2 d^3x,$$

where $\sum_{\ell}$ and $\sum_{n\ell}$ denote summation over all regions of linearity and nonlinearity respectively.[1] In analogy to the neutral particle case this expression may be reduced, with the aid of (VIII-2)[2], to

$$L = \sum_{n\ell} \frac{1}{16\pi} \int_{V_{n\ell}} (\phi^4 - 1) d^3x - \frac{1}{2}\, ab.$$

Again, only those solutions for which $\frac{dL}{db} = 0$ and the corresponding mass is positive, are taken to represent charged particles.

---

1 Recall (Chapter III) that the innermost region must be a nonlinear one while the outermost region ($r \geq b/\alpha$) must be a linear one for charged particles.

2 Note that $F(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 - \frac{1}{4}$ for charged particles in nonlinear regions.

A summary of the first few (numerically obtained) solutions is given below[1]:

group	L	b	a	M	$\phi(0)$[2]
(2,2;L)	21.13	8.01	-3.234	4.77	-18.63
(3,3;L,NL)	57.15	15.69	-4.246	9.48	35.69
(3,3;NL,L)	57.09	15.66	-4.238	9.30	35.77
(3,3;L,L)	57.09	15.67	-4.238	9.31	35.76

These "compound" solutions differ very little from those listed on p. 71 ; hence they are not separately plotted. As an example the solution (3,3;NL,L) is plotted in figure 7.6.

The search for nonspherical charged particlelike solutions was confined only to the possible existence of charged particles with a pure electric dipole moment; that is, well behaved solutions of:

---

1 (3,3;NL,L) refers to a solution belonging to the group (3,3) with the region about its first zero being nonlinear while that around its second zero being linear. The group (1,3,L) contains no particlelike solution.

2 L and M are in units of $\frac{mc^3\hbar}{g^2}$, a and $\phi(0)$ are in units of $\frac{mc^2}{g}$ and b is in units of $\frac{\hbar c}{g}$.

"Compound" Charged Particlelike Solution (3,3; NL, L)

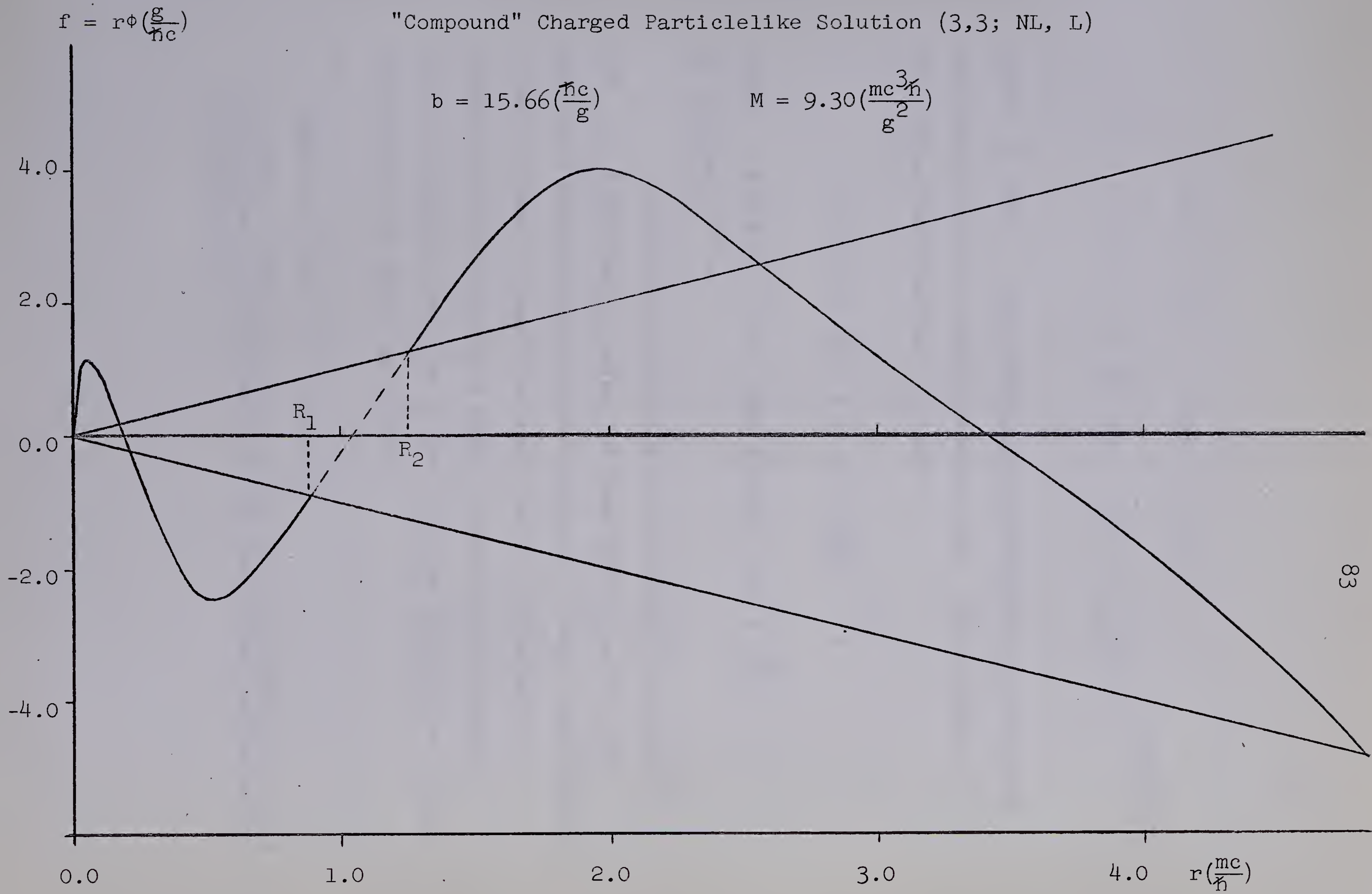
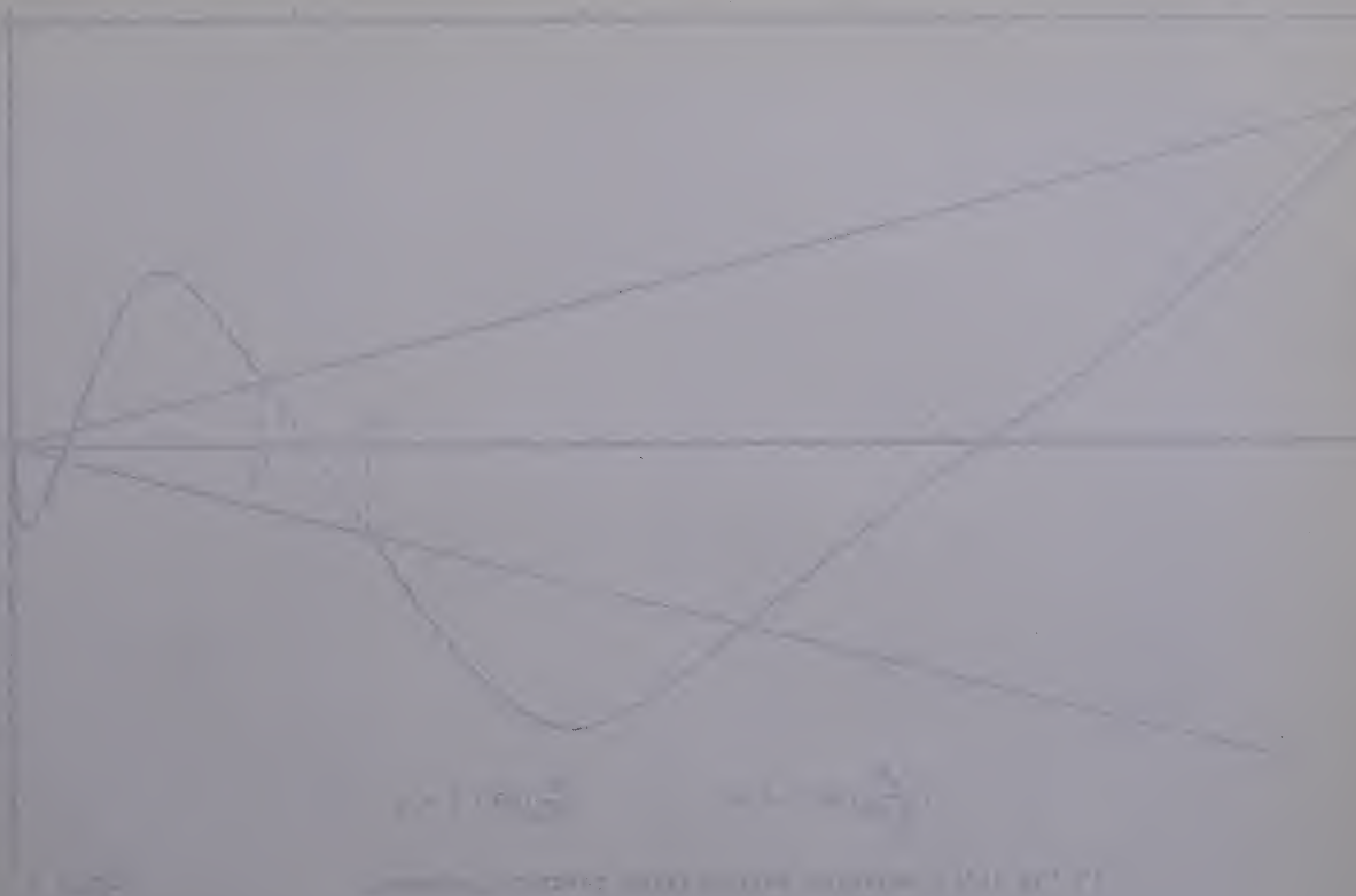


Fig. 7.6



$$\Delta\phi = \phi - \phi^3 \qquad \text{when} \quad r \leq R(\mu)$$

$$\phi = \pm 1 - \alpha + \frac{b}{r} + \frac{p\mu}{r^2} \qquad \text{for} \quad r \geq R(\mu), \tag{VII-13}$$

where b and p are constants proportional to the charge and electric dipole moment respectively. The surface of the charge distribution is given by

$$r = R(\mu) = \frac{b}{2\alpha} \left[1 + \left(1 + \frac{4p\alpha\mu}{b^2}\right)^{1/2} \right]$$

where $\mu \equiv \cos\theta$; for a real surface $|p| < \left|\frac{b^2}{4\alpha}\right|$ is required.

Attempts at obtaining well behaved solutions of (VII-13) numerically, using the same method as for non-spherical neutral particlelike solutions,[1] were unsuccessful.[2] It may be that no solutions exist which correspond to charged particles having a pure electric dipole moment, since in general the form of the solution outside the charge distribution is given by (III-7).

---

1 Chapter V, pp. 42-46 inclusive.

2 A sample result, for a = -2.177, b = 2.77 and p = .1043 is plotted (in cross-section for some fixed values of r) in Appendix R.

VIII. INTEGRAL RELATIONS SATISFIED BY WELL-BEHAVED SOLUTIONS

In previous chapters it was shown that well behaved, integrable solutions of (V-1) satisfy various integral relations such as (V-4), (V-6) and (V-7). These integral relations were obtained with the aid of the Lagrangian (V-2) which is extremised by the solutions of (V-1). It is apparent, however, that these (and other) properties of the solutions should be obtainable from (V-1) without recourse to a Lagrangian. This indeed is the case.

The procedure for obtaining integral relations which are satisfied by solutions of (V-1) consists, in essence, of integrating partially the identity[1]

$$\int_V \phi \xi_i{}^n \frac{\partial^n}{\partial \xi_i{}^n} (\Delta_\xi \phi) d^3\xi = \int_V \phi \xi_i{}^n \frac{\partial^n}{\partial \xi_i{}^n} F'(\phi) d^3\xi \qquad \text{(VII-1)}$$

for $i = 1,2,3$ and $n = 1, 2, \ldots$. In (VIII-1), (ξ_1, ξ_2, ξ_3) is a set of coordinates spanning V and Δ_ξ is the Laplacian operator expressed in these coordinates.

1 The use of this identity (which follows from (V-1)) is suggested by the work of Morgan and Ladsberg on hypervirial theorems (20).

The identity holds for any closed region V of space in which the solutions exist and are integrable (in the sense that all indicated integrals exist). Usually V will be taken to be all space.

In this chapter we derive some of the integral relations which are obtainable when the ξ_i are taken to be rectangular cartesian and spherical polar coordinates. First of all we note that all integral relations obtainable from (VIII-1) with $n = 0,1,2,\ldots,k$ are also obtainable from (VIII-1) with $n = k + 1$, since the "order" n of the relation may be reduced by one by partially integrating both sides of (VIII-1) once. For the sake of simplicity, however, we start with $n = 0$: In this case (VIII-1) implies, simply

$$\int_V \Delta\phi \;\; \phi \, d^3x = \int_V \phi \; F'(\phi) \; d^3x \quad ,$$

or

$$\int [\nabla\cdot(\phi\nabla\phi) - (\nabla\phi)^2] d^3x = \int \phi F'(\phi) \; d^3x \; .$$

Using the divergence theorem[1], we arrive at

$$\oint_S \phi \; \nabla\phi\cdot d\vec{s} - \int_V (\nabla\phi)^2 d^3x = \int_V \phi \; F'(\phi) d^3x \qquad \text{(VIII-2)}$$

---

1 Throughout this Chapter whenever the integral theorems of Gauss and Green are used it will be assumed that the solutions of (V-1) and their indicated derivatives are well behaved, so that these integral theorems are applicable.

where S is the closed surface bounding V. In particular, if ϕ or $\nabla\phi$ vanish on S, (VIII-2) becomes (V-6), while for charged particlelike solutions of (VII-1), (VIII-2) reduces to (VII-12).

If ξ_i is taken to be x, one of the rectangular cartesian coordinates then, for n = 1, (VIII-1) is

$$\int_V \phi x \frac{\partial}{\partial x} (\Delta\phi) d^3x = \int_V \phi x \frac{\partial}{\partial x} F'(\phi) d^3x ,$$

or

$$\int_V \phi x \, \Delta\left(\frac{\partial\phi}{\partial x}\right) d^3x = -\int_V x \frac{\partial}{\partial x} [F(\phi) - \phi \, F'(\phi)] \, d^3x.$$

Using Green's theorem on the left and integrating partially on the right[1]:

$$\int_V \frac{\partial\phi}{\partial x} \Delta(x\phi) d^3x + \int_S \left[x \, \phi \, \nabla\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial\phi}{\partial x} \nabla(x\phi)\right] \cdot d\vec{s}$$

$$= - \int \Big[x \, \{F(\phi) - \phi \, F'(\phi)\}\Big]_{x_1}^{x_2} dydz + \int_V [F(\phi) - \phi \, F'(\phi)] \, d^3x \qquad \text{(VIII-3)}$$

---

1 We assume, for definiteness, that $x_1 \leq x \leq x_2$ in V.

But

$$\int_V \frac{\partial\phi}{\partial x}\,\Delta(x\phi)d^3x = \int_V \frac{\partial\phi}{\partial x}\,(x\Delta\phi + 2\,\frac{\partial\phi}{\partial x})d^3x$$

$$=\int_V [x\frac{\partial}{\partial x}F(\phi) + 2(\frac{\partial\phi}{\partial x})^2\,]d^3x \text{ since } \Delta\phi=F'(\phi),$$

$$=\int\,[xF(\phi)]_{x_1}^{x_2}\,dydz + \int[2(\frac{\partial\phi}{\partial x})^2 - F(\phi)]\,d^3x,$$

after integration by parts.

This reduces (VIII-3) to

$$2\int_V(\frac{\partial\phi}{\partial x})^2\,d^3x + \int_S[x\phi\nabla(\frac{\partial\phi}{\partial x}) - \frac{\partial\phi}{\partial x}\,\nabla(x\phi)]\cdot d\vec{s} \qquad \text{(VIII-4)}$$

$$= \int\,[x\,\{\phi F'(\phi) - 2F(\phi)\}\,]_{x_1}^{x_2}\,dy\,dz + \int_V[2F(\phi)-\phi F'(\phi)]\,d^3x \quad .$$

Similar expressions hold for y and z. For localised[1] solutions, (VIII-4) reduces to (V-7), while for charged particlelike solutions of (VII-1) it becomes (VII-11).

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1 Throughout this chapter a "localised solution" is one which vanishes rapidly enough as $r \to \infty$, so that all indicated surface integrals vanish. Thus neutral particlelike solutions of (III-4) are "localised" in this sense, while the charged particlelike solutions (p. 65) are not.

For $n = 2$, (VIII-1) takes on the form

$$\int_V \phi x^2 \frac{\partial^2}{\partial x^2} (\Delta\phi)\, d^3x = \int_V \phi\, x^2 \frac{\partial^2}{\partial x^2} F'(\phi)\, d^3x$$

or

$$\int_V \phi x^2 \Delta \left(\frac{\partial^2\phi}{\partial x^2}\right) d^3x = \int_V \phi x^2 \left[F'''(\phi)\left(\frac{\partial\phi}{\partial x}\right)^2 + F''(\phi)\frac{\partial^2\phi}{\partial x^2}\right] d^3x \;.$$

Using Green's theorem on the left, we get:

$$\int_V \frac{\partial^2\phi}{\partial x^2}\left(x^2\Delta\phi + 4x\frac{\partial\phi}{\partial x} + 2\phi\right) d^3x + \oint_S \left[\phi x^2 \nabla\left(\frac{\partial^2\phi}{\partial x^2}\right) - \frac{\partial^2\phi}{\partial x^2}\nabla(\phi x^2)\right] \cdot d\vec{s}$$

$$= \int_V \left[\phi x^2\; F'''(\phi)\;\left(\frac{\partial\phi}{\partial x}\right)^2 + F''(\phi)\;\frac{\partial^2\phi}{\partial x^2}\right] d^3x \;.$$

Since $\Delta\phi = F'(\phi)$ this becomes

$$\int_V\left[2x\, \frac{\partial}{\partial x}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2\right\} + 2\phi\frac{\partial^2\phi}{\partial x^2}\right] d^3x + \oint_S \left[\phi x^2\; \nabla\left(\frac{\partial^2\phi}{\partial x^2}\right) - \frac{\partial^2\phi}{\partial x^2}\; \nabla\, (\phi x^2)\right] \cdot d\vec{s}$$

$$= \int x^2\left[\;(\phi F'' - F')\;\frac{\partial^2\phi}{\partial x^2} + F'''\cdot\left(\frac{\partial\phi}{\partial x}\right)^2\phi\;\right] d^3x$$

or

$$= \int x^2\; \frac{\partial^2}{\partial x^2}\;[\phi\; F' \;-\; 2F]\; d^3x \qquad .$$

Integrating partially on the left (once) and on the right (twice) we arrive at

$$2 \int \left[x\left(\frac{\partial\phi}{\partial x}\right)^2 + \phi\frac{\partial\phi}{\partial x}\right]_{x_1}^{x_2} dydz - 4 \int \left(\frac{\partial\phi}{\partial x}\right)^2 d^3x + \oint_S \left[\phi x^2 \nabla\left(\frac{\partial^2\phi}{\partial x^2}\right) - \frac{\partial^2\phi}{\partial x^2} \nabla(\phi x^2)\right] \cdot d\vec{s} \qquad \text{(VIII-5)}$$

$$= \int \left[x^2 \frac{\partial}{\partial x} (\phi F' - 2F) - 2x(\phi F' - 2F)\right]_{x_1}^{x_2} dydz + 2 \int (\phi F' - 2F) d^3x \; .$$

For localised solutions of (V-1) and for the charged particle-like solutions of (VII-1), (VIII-5) is identical to (VIII-4).

Henceforth we consider only localised, global solutions. For $n = 3$, (VIII-1) may be written as

$$\int \phi x^3 \Delta\left(\frac{\partial^3\phi}{\partial x^3}\right) d^3x = \int x^3 \phi \left[\left(\frac{\partial\phi}{\partial x}\right)^3 F^{IV} + 3\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial^2\phi}{\partial x^2}\right) F''' + \frac{\partial^3\phi}{\partial x^3} F''\right] d^3x \; .$$

Using Green's theorem on the left, and $\Delta\phi = F'(\phi)$ we get

$$6 \int x^2 \left(\frac{\partial^3\phi}{\partial x^3}\right)\left(\frac{\partial\phi}{\partial x}\right) d^3x + 6 \int x \frac{\partial^3\phi}{\partial x^3} \phi d^3x =$$

$$\int x^3 \left[\phi\left(\frac{\partial\phi}{\partial x}\right)^3 F^{IV} + 3\phi \frac{\partial\phi}{\partial x} \frac{\partial^2\phi}{\partial x^2} F''' + (\phi F'' - F')\frac{\partial^3\phi}{\partial x^3}\right] d^3x \; .$$

Integrating partially on the left:

$$-6 \int \frac{\partial^2 \phi}{\partial x^2} \left(x^2 \frac{\partial^2 \phi}{\partial x^2} + 2x \frac{\partial \phi}{\partial x}\right) d^3x - 6 \int \frac{\partial^2 \phi}{\partial x^2} \left(\phi + x \frac{\partial \phi}{\partial x}\right) d^3x$$

$$= \int x^3 \left[\frac{\partial^3}{\partial x^3} (\phi F' - 2F) - \left(\frac{\partial \phi}{\partial x}\right)^3 F'''\right] d^3x ,$$

or

$$6 \int x^2 \left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 d^3x + 9 \int x \frac{\partial}{\partial x} \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x}\right)^2\right] d^3x + 6 \int \phi \frac{\partial^2 \phi}{\partial x^2} d^3x$$

$$= 6 \int (\phi F' - 2F) d^3x + \int x^3 \left(\frac{\partial \phi}{\partial x}\right)^3 F''' d^3x .$$

Further partial integration of the integrals on the left implies that

$$6 \int x^2 \left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 d^3x - 15 \int \left(\frac{\partial \phi}{\partial x}\right)^2 d^3x = 6 \int (\phi F' - 2F) d^3x +$$

$$\int x^3 \left(\frac{\partial \phi}{\partial x}\right)^3 F''' d^3x .$$

Finally, using (VIII-4), we obtain the relation

$$\int \left(\frac{\partial \phi}{\partial x}\right)^2 d^3x = 2 \int \left[x^2 \left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 - \frac{1}{6} x^3 \left(\frac{\partial \phi}{\partial x}\right)^3 F'''\right] d^3x , \quad \text{(VIII-6)}$$

and similar expressions for y and z.

We consider now integral relations satisfied by localised, global solutions of (V-1), when ξ_i is taken to be r, the radial spherical polar coordinate. When $n = 0$, we obtain the relation (V-6), and when $n = 1$ or 2, the relation (V-4) which also follows from (VIII-4). In any case no new relations are obtained when $n = 0, 1, 2$.

For $n = 3$, (VIII-1) becomes

$$\int \phi r^3 \frac{\partial^3}{\partial r^3} (\Delta\phi)\, d^3x = \int \phi r^3 \frac{\partial^3}{\partial r^3} F'(\phi)\, d^3x : \qquad \text{(VIII-7)} .$$

But

$$\frac{\partial^3}{\partial r^3} (\Delta\phi) = \Delta\left(\frac{\partial^3\phi}{\partial r^3}\right) - \frac{6}{r} \Delta \left(\frac{\partial^2\phi}{\partial r^2}\right) + \frac{18}{r^2} \Delta\left(\frac{\partial\phi}{\partial r}\right) - \frac{24}{r^3} \Delta\phi$$

$$+ \frac{6}{r} \frac{\partial^4\phi}{\partial r^4} - \frac{12}{r^2} \frac{\partial^3\phi}{\partial r^3} + \frac{36}{r^4} \frac{\partial\phi}{\partial r} . \quad [1]$$

Making use of Green's theorem, we find that

$$\int \phi r^3 \Delta\left(\frac{\partial^3\phi}{\partial r^3}\right) d^3x = \int \frac{\partial^3\phi}{\partial r^3} \left(r^3\Delta\phi + 6r^2 \frac{\partial\phi}{\partial r} + 12\, r\phi\right) d^3x,$$

$$\int \phi r^2 \Delta\left(\frac{\partial^2\phi}{\partial r^2}\right) d^3x = \int \frac{\partial^2\phi}{\partial r^2} \left(r^2\Delta\phi + 4r \frac{\partial\phi}{\partial r} + 6\phi\right) d^3x ,$$

---

1 This is shown in Appendix S.

and

$$\int \phi r\Delta \left(\frac{\partial\phi}{\partial r}\right) d^3x = \int \frac{\partial\phi}{\partial r} \left(r\Delta\phi + 2\,\frac{\partial\phi}{\partial r} + \frac{2}{r}\,\phi\right) d^3x \ .$$

Thus, (VIII-7) becomes

$$\int \Big[6r^2 \frac{\partial^3\phi}{\partial r^3}\frac{\partial\phi}{\partial r} - 24r\,\frac{\partial^2\phi}{\partial r^2}\frac{\partial\phi}{\partial r} - 36\,\phi\,\frac{\partial^2\phi}{\partial r^2} + 36\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{72}{r}\,\phi\frac{\partial\phi}{\partial r} +$$

$$6\,r^2\phi\,\frac{\partial^4\phi}{\partial r^4} \quad +$$

$$\left\{r^3\,\frac{\partial^3\phi}{\partial r^3} - 6r^2\,\frac{\partial^2\phi}{\partial r^2} + 18r\,\frac{\partial\phi}{\partial r} - 24\phi\right\}\Delta\phi\,\Big] d^3x =$$

$$\int r^3\,\phi\,\left[\left(\frac{\partial\phi}{\partial r}\right)^3 F^{IV} + 3\,\frac{\partial\phi}{\partial r}\,\frac{\partial^2\phi}{\partial r^2}\,F''' + \frac{\partial^3\phi}{\partial r^3}\,F''\right] d^3x \ ,$$

or

$$\int \Big[6r^2 \frac{\partial^3\phi}{\partial r^3}\frac{\partial\phi}{\partial r} - 24r\,\frac{\partial^2\phi}{\partial r^2}\frac{\partial\phi}{\partial r} - 36\phi\,\frac{\partial^2\phi}{\partial r^2} + 36\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{72}{r}\,\phi\,\frac{\partial\phi}{\partial r}$$

$$+ 6\,r^2\phi\,\frac{\partial^4\phi}{\partial r^4} \ - 6\left(r^2\frac{\partial^2\phi}{\partial r^2} - 3r\,\frac{\partial\phi}{\partial r} + 4\phi\right)\Delta\phi\,\Big] d^3x \qquad \text{(VIII-8)}$$

$$= \int r^3\,\left[\phi\,\left(\frac{\partial\phi}{\partial r}\right)^3 F^{IV} + 3\phi\,\frac{\partial\phi}{\partial r}\,\frac{\partial^2\phi}{\partial r^2}\,F''' + (\phi F'' - F')\frac{\partial^3\phi}{\partial r^3}\right] d^3x \ .$$

Integrating by parts:

$$\int r^2 \left(\frac{\partial^3\phi}{\partial r^3}\frac{\partial\phi}{\partial r} + \phi \frac{\partial^4\phi}{\partial r^4}\right)d^3x = \int r^2 \frac{\partial}{\partial r}\left(\phi\frac{\partial^3\phi}{\partial r^3}\right)d^3x = -4\int r\phi \frac{\partial^3\phi}{\partial r^3}\, d^3x,$$

$$\int r\left(\frac{\partial^2\phi}{\partial r^2}\frac{\partial\phi}{\partial r} + \phi\frac{\partial^3\phi}{\partial r^3}\right)d^3x = \int r \frac{\partial}{\partial r}\left(\phi \frac{\partial^2\phi}{\partial r^2}\right)d^3x = -3\int \phi\frac{\partial^2\phi}{\partial r^2}\, d^3x$$

and

$$\int \frac{1}{r}\phi \frac{\partial\phi}{\partial r}\, d^3x = \frac{1}{2}\int\phi \frac{\partial\phi}{\partial r}\, d(r^2)d\Omega = -\frac{1}{2}\int \left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \phi\frac{\partial^2\phi}{\partial r^2}\right] d^3x$$

where $r^2 dr d\Omega \equiv d^3x$. Thus (VIII-8) reduces to

$$-6\int \left[r^2 \frac{\partial^2\phi}{\partial r^2} - 3r\frac{\partial\phi}{\partial r} + 4\phi\right]\Delta\phi \quad d^3x = \int r^3\left[\frac{\partial^3}{\partial r^3}(\phi F' - 2F) - r^3\left(\frac{\partial\phi}{\partial r}\right)^3 F'''\right]d^3x.$$

Using $\Delta\phi = F'(\phi)$ and integrating partially on the right,

$$\int \left[r^2 \frac{\partial^2\phi}{\partial r^2} - 3r\frac{\partial\phi}{\partial r} + 4\phi\right] F'(\phi)\ d^3x = \int \left[10(\phi F' - 2F) + \frac{1}{6} r^3\left(\frac{\partial\phi}{\partial r}\right)^3 F'''\right]d^3x.$$

Finally, using (VIII-4), we obtain the relation:

$$\frac{1}{3}\int (\nabla\phi)^2 d^3x = \frac{1}{20}\int\left[\frac{1}{6} r^3\left(\frac{\partial\phi}{\partial r}\right)^3 F''' - \left(r^2 \frac{\partial^2\phi}{\partial r^2} - 3r\frac{\partial\phi}{\partial r} + 4\phi\right)F'\right]d^3x. \tag{VIII-9}$$

For the particular case $F'(\phi) = \phi - \phi^3$, (V-4), (V-6) and some further partial integration reduces (VIII-9) to:[1]

$$\frac{1}{3} \int (\nabla\phi)^2 d^3x = \frac{2}{5} \int [\phi(\frac{\partial\phi}{\partial r})^3 r^3 + r^2(3\phi^2-1)(\frac{\partial\phi}{\partial r})^2] d^3x \quad .(\text{VIII-10})$$

Still other integral relations may be obtained from (VIII-1) by using higher values of n and other coordinates ξ_i.

Some uses of the integral relations have already been demonstrated in previous chapters. They may be used, for example, to test approximate solutions of $\Delta\phi = F'(\phi)$. Thus for neutral particlelike solutions of $\Delta \phi = \phi - \phi^3$, the integral relations (V-4), (V-6), (VIII-6), (V-7) and (VIII-10) imply that the energy of the system represented by a solution ϕ:

$$M = \frac{1}{4\pi} \int [\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} \phi^2 - \frac{1}{4} \phi^4] d^3x$$

may be equivalently expressed by

$$E = \frac{1}{4\pi} \int \phi^2 d^3x = \frac{1}{16\pi} \int \phi^4 d^3x = \frac{1}{12\pi} \int (\nabla\phi)^2 d^3x, \quad (\text{VIII-11})$$

$$E_x = \frac{1}{4\pi} (\frac{\partial\phi}{\partial x})^2 d^3x \qquad \text{and similarly for y and z,}$$

---

1 Note that (VIII-10) does not follow from (VIII-6), but is (in general) an independent relation.

$$E_r = \frac{1}{4\pi}\frac{2}{5}\int[\phi \left(\frac{\partial\phi}{\partial r}\right)^3 r^3 + r^2(3\phi^2 -1)\left(\frac{\partial\phi}{\partial r}\right)^2]d^3x ,$$

or

$$E_{xx} = \frac{2}{4\pi}\int [x^2 \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + x^3 \phi \left(\frac{\partial\phi}{\partial x}\right)^3]d^3x$$

and similar expressions for y and z.

In table VIII A, we list some of these quantities for various variational approximations to the two lowest (in energy) spherically symmetric neutral particlelike solutions. These variational approximations were obtained by Betts, Schiff and Strickfadden (10).

In table VIII B we list similar results for various variational approximations to possible axially symmetric, odd parity neutral particlelike eigensolutions. The indicated approximate variational solutions are those which were listed previously in tables VI A and VI B.

In these tables all results are in units of $\frac{mc^3\hbar}{g^2}$. The results for the nonspherical functions were obtained in part by numerical integration. The symbol E, where it appears, implies that the corresponding integral relation is identically satisfied by the indicated trial function. For example, in table VIII B, E appears under E_x and E_z for the function $Az' e^{-r'}$. This means, simply, that this

trial function satisfies identically the integral relation

$$E = E_x = E_z \quad .$$

It is interesting to note, in table VIII A, that if ΔE represents the difference (in absolute value) between the values of E of a given trial function and the "exact" solution, then the corresponding deviations for the other integrals (that is ΔE_x, ΔE_z, etc.) are at most one order greater than ΔE. For example for the trial function 2 $\Delta E \simeq .04$ while $\Delta E_r \simeq .12$ and $\Delta E_{xx} \simeq .08$ and for the function 5, $\Delta E \simeq .51$ while $\Delta E_r \simeq 1.36$ and $\Delta E_{xx} = 1.46$. This is consistent with the fact that the trial functions extremise E but not (necessarily) E_x, E_z, etc.

It is not clear whether the results for the odd parity states exhibit such a consistency. As pointed out in Chapter VI some of these variational trial functions, especially ϕ_B, ϕ_C and ϕ_E, seem to approximate a two-particle state rather than a one-particle state of odd parity. It is clear, also, that non-spherical solutions are much more difficult to approximate variationally than spherical ones, hence these approximate results may still be appreciably different from the exact solutions. As a result, a trial function, which gives a reasonable approximation to the energy E, may still give poor results for the other integrals, which are not extremised in the variation.

Table VIIIA

Integral Relations of Variational Trial Functions for Spherically Symmetric Neutral Particlelike Solutions.

Trial Function	E	E_x	E_z	E_r	E_{xx}
A. Ground State					
1. "Exact" solution (numerically obtained)	1.503	E	E	E	E
2. $\lambda e^{-\alpha r}$, $\lambda=4\sqrt{2}$, $\alpha = \sqrt{3}$	1.54	E	E	1.38	1.43
3. $A[\frac{1}{r}(e^{-r}-e^{-\alpha r})+ be^{-\alpha r}]$, $A = 2.7094$ $\alpha = 5.491$ $b = -2.95$	1.505	E	E	1.499	1.489
B. First "Excited" State					
4. "Exact" solution (numerically obtained)	9.46	E	E	E	E
5. $A(1+cr)e^{-\alpha r}$, $A = 14.38$, $c= -1.876$, $\alpha = 1.802$	9.97	E	E	10.82	10.92

Table VIII B

Integral Relations of Variational Trial Functions for Possible Odd Parity Neutral Particlelike Solutions.

Trial Function	E	E_x	E_z	E_r	E_{xx}	E_{zz}
<u>A. State with lowest energy</u>						
6. $Aze^{-\alpha r}$	5.47	3.28	13.12	4.38	.512	7.83
7. $Az'e^{-r'}$	4.41	E	E	3.53	4.69	-4.28
8. $A(P_1(\mu)r'e^{-r'}+P_3(\mu)r'^3e^{-\eta r'}+bP_5(\mu)r'^5e^{-\sigma r'})$	3.96	E	E	2.96	4.75	-3.17
9. $\phi_A^{(1)} = R^{(1)}(r)\ P_1(\mu)$	4.87	2.92	8.76	E		
10. $\phi_B^{(1)} = r^n e^{-\alpha r}\ Y^{(1)}(\mu)$	3.77	2.00	7.31	2.96		
11. $\phi_C^{(1)} = R^{(1)}(r)\ P_1(\mu) + Q^{(1)}(r)\ P_3(\mu)$	3.63	2.91	5.08	E		
12. $\phi_E^{(1)} = R^{(1)}(r)\ P_1(\mu) + Q^{(1)}(r)\ P_3(\mu) + S^{(1)}(r)\ P_5(\mu)$	3.31	2.94	4.04	E		

Table VIII B

(continued)

	Trial Function	E	E_x	E_z	E_r
B.	Various possible "excited" states				
13.	$\phi_A^{(2)} = R^{(2)}(r)P_1(\mu)$	16.60	9.96	29.87	E
14.	$\phi_A^{(3)} = R^{(3)}(r)P_1(\mu)$	37.88	22.73	68.05	E
15.	$\phi_B^{(2)} = r^n\ e^{-\alpha r}\ Y^{(2)}(\mu)$	31.33	28.11	37.80	21.98
16.	$\phi_B^{(3)} = r^n\ e^{-\alpha r}\ Y^{(3)}(\mu)$	109.8	100.4	128.9	77.15
17.	$\phi_C^{(2)} = R^{(2)}(r)P_1(\mu) + Q^{(2)}(r)P_3(\mu)$	10.72	7.23	17.70	E
18.	$\phi_C^{(3)} = R^{(3)}(r)P_1(\mu) + Q^{(3)}(r)P_3(\mu)$	26.75	23.94	32.36	E
19.	$\phi_C^{(4)} = R^{(4)}(r)P_1(\mu) + Q^{(4)}(r)P_3(\mu)$	38.92	26.76	63.25	E

IX. STABILITY OF PARTICLELIKE SOLUTIONS

The question of stability of particlelike solutions of

$$\Delta\phi = F'(\phi) \tag{IX-1}$$

has been considered by Derrick (18) who proved (as already mentioned in Chapter V) that the integrable solutions are unstable with respect to radial perturbations. Rosen (21) then proved that Derrick's instability criterion[1] implies, for a wide class of field equations,[2] that the solutions are also dynamically unstable in the more general sense of Liapunov.

In this chapter we consider the dynamical stability of particlelike solutions of the field equations (I-1). More specifically we shall consider only the dynamical stability of the neutral particlelike solutions discussed in Chapter VI. These solutions are not stable in the sense of Derrick's theorem,

---

1 vis $\left.\dfrac{\partial^2 L\,[\phi(\mu r)]}{\partial\mu^2}\right|_{\mu=1} < 0$ for all integrable solutions $\phi(r)$.

2 i.e., those derivable from the Lagrangian density

$$\mathcal{L} = (\nabla\phi)^2 - \left(\frac{\partial\phi}{\partial t}\right)^2 + F'(\phi).$$

however their (physically more meaningful) dynamical stability is not immediately precluded. This is because the Lagrangian (I-2) does not fall within the class $\mathcal{L} = (\nabla\phi)^2 - (\frac{\partial\phi}{\partial t})^2 + F'(\phi)$ mentioned in the footnote on p. 101, even when $\vec{H} = 0$. [1]

Actually, one might expect the neutral particle-like solutions to be unstable, particularly so since, as pointed out by Rosen (21), all spinless "elementary" particles (such as π, K) known to date are only "metastable" (that is, their decay rate is $<< \frac{mc^2}{\hbar}$.

To determine the dynamical stability of a neutral particlelike solution $\phi_0(r)$, we consider a general time-dependent perturbation of this solution:

$$A_i \equiv [\vec{0}, i\ \phi_0(r)] \rightarrow A_i + \delta A_i \equiv [\vec{a}(\vec{r},t), i\ \{\phi_0(r) + \psi(\vec{r},t)\}\ ] .$$

We assume that the perturbations $\vec{a}$ and ψ are initially small. Substituting the perturbed expressions into the field equations (I-1a), we get (to first order of small quantities):

---

1 In fact, the field equations (I-1) imply that when $\vec{A} = 0$, $\frac{\partial\phi}{\partial t} = 0$, that is only static solutions are possible.

$$\nabla \times \nabla \times \vec{a} + \frac{\partial^2 \vec{a}}{\partial t^2} + \nabla\left(\frac{\partial \psi}{\partial t}\right) = (\phi_0^{\,2} - 1)\,\vec{a}$$

$$\Delta\psi + \frac{\partial}{\partial t}\,(\nabla\cdot\vec{a}) = (1 - 3\phi_0^{\,2})\psi \qquad \text{(IX-2)}$$

From physical considerations we expect the particles to decay into other particles or/and radiation.

Consider first the possibility that the particle decays into (plane wave) radiation directly. Thus, we assume that the (initially small) perturbation is plane wave-like only.[1] In general, then:

$$\vec{a} = \sum_k c_k\, \vec{a}_k\, (\sigma_k)$$

and

$$\psi = \sum_k c_k\, \psi_k\, (\sigma_k)$$

where $\sigma_k = \vec{k}\cdot\vec{r} - \omega t$ and the c_k are arbitrary but small constants. The summations are taken over all values of the four-vector $k \equiv (\vec{k}, i\omega)$.

Substituting these expressions into (IX-2) we get[2], by virtue of the independence of the constants c_k,

---

1 Note that this is not a localised perturbation.

2 Recall that $\nabla \times \nabla \times \vec{a} = \nabla(\nabla\cdot\vec{a}) - \Delta\vec{a}$.

$$\vec{k}(\vec{k}\cdot\vec{a}_k{}'') - k^2\vec{a}_k{}'' - \omega\vec{k}\,\psi_k{}'' - (\phi_0{}^2-1)\vec{a}_k = 0 \qquad \text{(IX-3)}$$

and

$$\vec{k}^2\psi_k{}'' - \omega\vec{k}\cdot\vec{a}_k{}'' + (3\,\phi_0{}^2-1)\psi_k = 0, \qquad \text{(IX-4)}$$

where the prime $\equiv \frac{d}{d\sigma_k}$ and $k^2 = \vec{k}^2 - \omega^2$. Since the coefficient of ψ_k in (IX-4) cannot be expressed as a function of σ_k only, we must have $\psi_k = 0$, hence (for nontrivial solutions) $\vec{k}\cdot\vec{a}_k{}'' = 0$. (IX-3) then reduce to

$$k^2\,\vec{a}_k{}'' + [\phi_0{}^2(r) - 1]\,\vec{a}_k = 0$$

which again contradicts our assumption that $\vec{a}_k = \vec{a}_k(\sigma_k)$ only. Thus, the initial perturbation cannot be purely plane wave-like, implying that it is impossible for a neutral particle to decay directly into plane wave radiation.

Consider now the possibility that the particle represented by $\phi_0(r)$ decays directly into another particle. Assume that the perturbations can be expanded in a series over the (complete) set of functions e^{ikt}:

$$\vec{a} = \sum_k \mathrm{Re}\ \vec{u}_k(\vec{r})\ c_k e^{ikt}\ ,$$

and $\qquad$ (IX-5)

$$\psi = \sum_k \mathrm{Re}\ v_k(\vec{r}) c_k e^{ikt}\ ,$$

where the c_k are arbitrary (small) constants. The summation is taken over all admissible values of k. Substituting (IX-5) into (IX-2), we find that $\vec{u}_k$ and v_k must satisfy:

$$\nabla \times \nabla \times \vec{u}_k - (k^2 + \phi_0^{\,2} - 1)\vec{u}_k + ik\,\nabla v_k = 0$$

$$\Delta v_k + (3\,\phi_0^{\,2} - 1)v_k + ik\,\nabla\cdot\vec{u}_k = 0 \quad . \qquad \text{(IX-6)}$$

Since the perturbations are particlelike, we require that $\vec{u}_k$ and v_k be everywhere finite,[1] and $\vec{u}_k$, v_k must $\to 0$ as $r \to \infty$ for a localised disturbance.[2]

The problem is thus to find the eigenvalues k of this set of coupled equations (IX-6), with the indicated boundary conditions. An imaginary eigenvalue, if one turned up, would imply a decay rate of $(-k^2)^{1/2}\,\frac{mc^2}{\hbar}$.

In general it appears rather difficult to say anything definite about the nature of the eigenvalue spectrum of k (e.g. whether discrete, real, etc.). We consider some

---

1 Also since the perturbations are assumed initially small.

2 This might be relaxed somewhat to requiring that $v_k \to$ a constant γ_k (for example for a charged particlelike perturbation).

special forms of the disturbance which reduce (IX-6) to a simpler form. Assume a perturbation for which $\nabla \times \vec{u}_k = 0$ that is $\vec{H}$ remains null. It then follows from (IX-6) that

$$\vec{u}_k = \frac{ik \, \nabla v_k}{(k^2 - 1 - \phi_0^{\,2})}$$

and

$$\Delta v_k \left[1 - \frac{k^2}{k^2 - 1 + \phi_0^{\,2}}\right] + \frac{2k^2 \phi_0 \phi_0'}{(k^2 - 1 + \phi_0^{\,2})^2} \frac{\partial v_k}{\partial r} + (3\phi_0^{\,2} - 1) v_k = 0$$

(IX-7)

Let $v_k = \sum_{\ell,m} V_{\ell m} \, w_k(r) Y_\ell^m(\theta,\varphi)$, $V_{\ell m}$ being arbitrary constants. Then, from (IX-7), we obtain[1]

$$\frac{d^2 w_k}{dr^2} + 2\left[\frac{1}{r} + \frac{k^2 \phi_0 \phi_0'}{(\phi_0^{\,2} - 1)(k^2 + \phi_0^{\,2} - 1)}\right] \frac{dw_k}{dr} + \left[\frac{(3\phi_0^{\,2})(k^2 - 1 + \phi_0^{\,2})}{\phi_0^{\,2} - 1} - \frac{\ell(\ell+1)}{r^2}\right] w_k$$

$$= 0 \qquad \text{(IX-7a)}$$

with $w_k(r) \to 0$ as $r \to \infty$

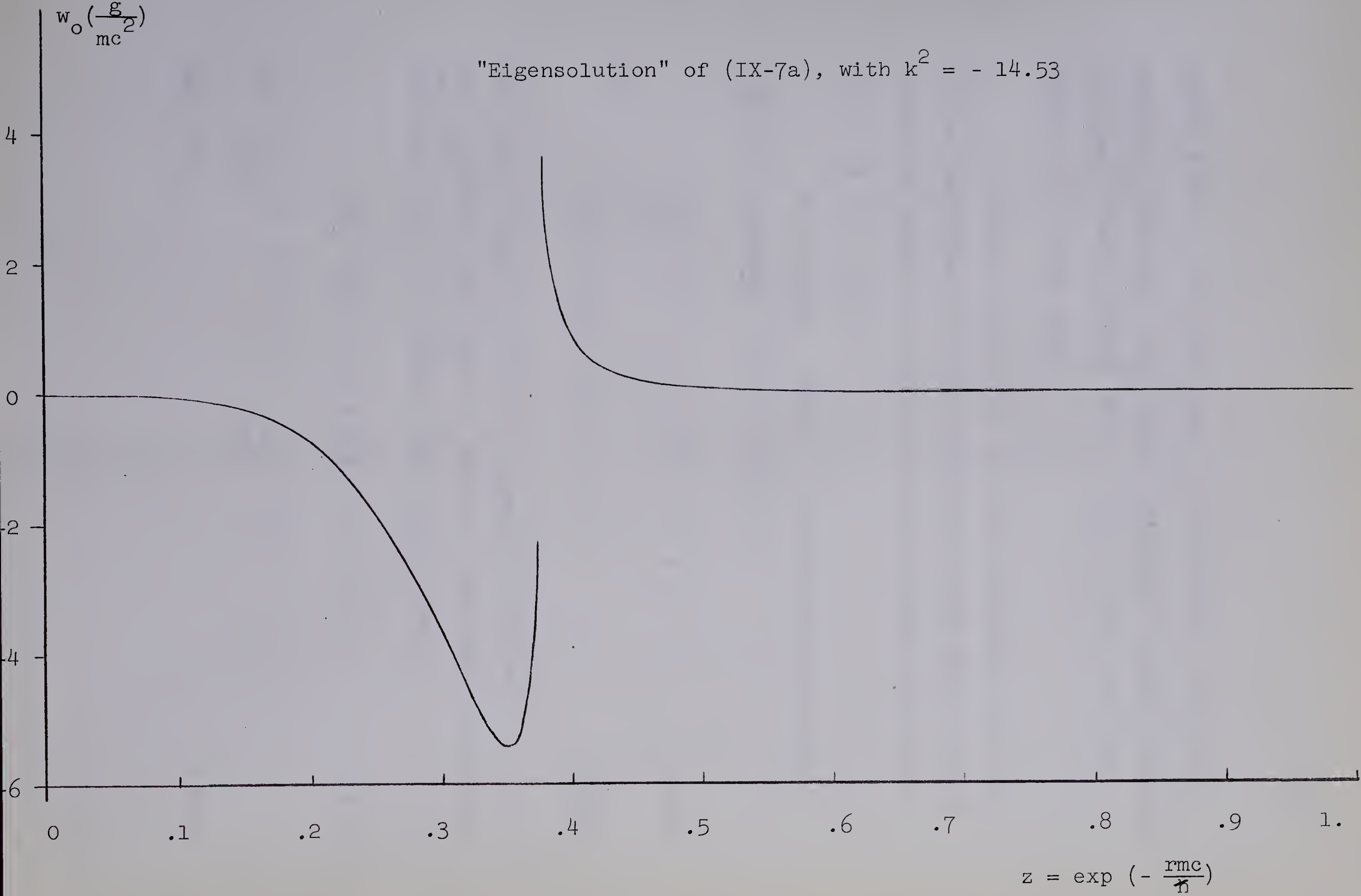
and $\lim_{r \to 0} \left[r \, w_k(r) \right] = 0$.

Approximate solutions of this equation were obtained numerically. One such eigensolution (with $k^2 = -14.53$ and $\ell = 0$) is plotted in figure 9.1. No well behaved eigensolutions were found,[2] thus contradicting our initial assumption of

---

1 $\phi_0' \equiv \dfrac{d\phi_0}{dr.}$

2 Note that the coefficients of the equation are singular at $\phi_0^{\,2} = 1$. The eigensolutions also appear to be singular there.



$w_o(\frac{g}{mc^2})$
"Eigensolution" of (IX-7a), with $k^2 = -14.53$
4
2
0
-2
-4
-6
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.
$z = \exp(-\frac{rmc}{\hbar})$

Fig. 9.1

possible $\vec{H} = 0$, initially small perturbations. It is not possible, therefore, for neutral particles to decay directly into particles for which $\vec{H} = 0$, that is they are stable against this decay mode.

Further inspection of (IX-6) indicates also that neutral particles are stable against a decay mode in which either the vector or scalar potentials remain unperturbed (i.e. $\vec{u}_k$ or $v_k = 0$).

We consider, finally, a perturbation for which $\nabla\cdot\vec{u}_k = 0$, whereupon (IX-6) reduce to

$$\Delta\vec{u}_k + (k^2 - 1 + \phi_0^{\,2})\vec{u}_k + ik\,\nabla\, v_k = 0 \qquad \text{(IX-8)}$$

and

$$\Delta v_k + (3\phi_0^{\,2} - 1)\, v_k = 0 \qquad \text{(IX-9)}$$

The latter equation, with the boundary conditions $\lim_{r\to\infty} v_k(\vec{r}) = 0$ and $\lim_{r\to 0} [r\, v_k(\vec{r})] = 0$ has only the trivial solution $v_k = 0$, in which case (IX-8) becomes

$$\Delta\vec{u}_k + (k^2 - 1 + \phi_0^{\,2})\vec{u}_k = 0 \qquad \text{(IX-10)}$$

with $\lim_{r\to\infty} \vec{u}_k(\vec{r}) = 0$, $\lim_{r\to 0} r\,\vec{u}_k(r) = 0$

and $\nabla\cdot\vec{u}_k = 0$ (IX-11)

A solution of (IX-11) is $\vec{u}_k(r) = u_k(r,\theta)\ \vec{L}_\varphi$, $\vec{L}_\varphi$ being a unit vector in the direction of increasing φ. (IX-10) then becomes

$$\Delta u_k + (K^2 + \phi_0{}^2 - \frac{1}{r^2 \sin^2\theta})u_k = 0 \qquad \text{(IX-12)}$$

where $K^2 = k^2 - 1$. (IX-12) can be simplified by variable separation, whereupon we find that

$$u_k(r,\theta) = \sum_{\ell=1}^{\infty} \frac{1}{r} f_{k\ell}(r)\ P_{\ell 1}\ (\cos\theta)\ ,$$

where $P_{\ell 1}$ are the associated Legendre functions $P_{\ell m}$ with $m = 1$, and $f_{k\ell}(r)$ are eigensolutions of

$$\frac{d^2 f_{k\ell}}{dr^2} + [k^2 + \phi_0{}^2(r) - \frac{\ell(\ell+1)}{r^2}]\ f_{k\ell} = 0 \qquad \text{(IX-13)}$$

with $\ell \geq 1$, $f_{k\ell}\ (0) = 0$ and $\lim_{r\to\infty} (\frac{f_{k\ell}}{r}) = 0$.
When $\ell = 0$, the equation (IX-13) has the solution $f_{ko} = r\ \phi_0$ and $K^2 = -1$ hence the ground state eigensolution of (IX-13) (when $\ell \geq 1$) is clearly $K_1{}^2 > -1$, that is $k^2 > 0$. It then follows that the (spherically symmetric) neutral particles are stable against this particular decay mode.

As remarked previously, it is not essential to consider localised particlelike perturbations only. Thus the boundary condition on (IX-6) at infinity can be generalised to:

$$\vec{u}_k, v_k \quad \rightarrow \quad \text{const.} \neq 0 \quad \text{as} \quad r \rightarrow \infty \quad ,$$

corresponding to a nonlocalised, but still particlelike and initially small (by virtue of the small coefficients c_k which appear in the expansions (IX-5)) perturbation. This, however, reduces to the localised situation, at least for the two special cases of (IX-6) which have been considered ($\nabla \times \vec{u}_k = 0$ and $\nabla \cdot \vec{u}_k = 0$). This is because for large r (when $\phi_0 \rightarrow 0$ as e^{-r}/r) (IX-7) and (IX-9) reduce to

$$\Delta v_k + (k^2 - 1) v_k = 0$$

and

$$\Delta v_k - v_k = 0 \quad \text{respectively, and}$$

the well behaved solutions of these equations are asymptotic to zero at infinity.

X. CONCLUSIONS

In this thesis we have been concerned, primarily, with the investigation of the properties of static, particlelike solutions (in the absence of the magnetic field) of the field equations proposed by Dr. Schiff. The only other types of solutions investigated at some length were plane-wavelike solutions. It was found that, in addition to the usual Maxwellian plane waves, the theory admits plane-wavelike solutions for which the electric field is not orthogonal to the direction of propagation of the wave and the charge and current densities are non-zero.

The existence of global, well behaved, spherically symmetrical solutions of $\Delta\phi = \phi - \phi^3$ was formally established. The existence of well behaved nonspherical solutions, however, was demonstrated only in the vicinity of an arbitrary point of space.

The variational method for obtaining approximations to well behaved solutions of the general class of field equations $\Delta\phi = F'(\phi)$ was considered and sufficient conditions were derived under which a (variational) upper bound to the Lagrangian is assured. These conditions assure upper bounds for all eigenstates (not only the one of lowest energy) without recoure to any other subsidiary conditions (such as orthogonality conditions). The usual technique of parameter variation was generalised, for the non-spherical case, to include variation of function of one of the independent variables.

The list of known spherically symmetric neutral and charged particlelike solutions of the field equations was extended to include "compound" particlelike solutions, which are linear in the vicinity of (some of) their zeros.

A variational principle for the charged particlelike solutions was determined and used to obtain approximations to the lowest (in energy) spherically symmetric charged particlelike state. It was found that this procedure is rather inefficient for obtaining approximations to the mass and charge of the particle, since it is the Lagrangian and not these quantities which is extremised in the variational procedure.

Approximation techniques, numerical and variational, were used in an attempt to determine non-spherical, axially symmetric, neutral particlelike solutions of odd parity. The results, unfortunately, do not unambiguously indicate the existence of neutral, one-particle, non-spherical states. The numerical results (obtained by iteratively solving the finite difference approximation to the field equation) could not, in any sense, be interpreted as representing well behaved solutions, since they were found to be highly unstable near the origin. The variational results on the other hand, seem to suggest a number of possibilities: The best results obtained by the parameter variation technique (table VI-A) suggest the existence of a one-particle, odd parity state of mass $\lesssim 3.96 \frac{mc^3\hbar}{g^2}$.

The nodeless variational trial functions, obtained by the variation of function method, (table VI-B), it is believed, approximate the lowest two-particle state which has an energy of 3.006 $\frac{mc^3\hbar}{g^2}$. It is possible, however, that they approximate a one particle odd parity state of energy not much above this value. The various trial functions with nodes (table VI-B), are approximations, in all probability, to various excited odd parity states.

A numerical search was also made for a (non-spherical) particlelike state corresponding to a charged particle with a pure electric dipole moment. No such solution was found.

A scheme for obtaining integral relations satisfied by integrable solutions of $\Delta\phi = F'(\phi)$ was developed and used to obtain the first few integral relations. These relations were then used to obtain various expressions for the energy corresponding to a given solution. They were used, also, to test variational approximations to the solutions of $\Delta\phi = \phi-\phi^3$. As might be expected, it was found that the variational approximations to the spherical states satisfy these integral relations much more closely than do approximations to odd parity states.

Finally, the dynamical stability of the spherically symmetric particlelike solutions was investigated. They were found to be stable against many dissociation modes, among them against dissociation into plane waves and particles with $\vec{H} = 0$.

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APPENDIX A: COMPONENTS OF ENERGY-MOMENTUM TENSOR

In vectorial notation, the components of the symmetrised energy-momentum tensor (I-3), are:

$$T_{\alpha\beta} = \frac{N}{4\pi} \quad [\tfrac{1}{2}(\vec{E}^2+\vec{H}^2)\delta_{\alpha\beta} - E_\alpha E_\beta - H_\alpha H_\beta - \{\tfrac{1}{2}(\vec{A}^2-\phi^2) + \tfrac{1}{4}(\vec{A}^2-\phi^2)^2 + K\}\delta_{\alpha\beta}$$

$$+ \quad (1 + \vec{A}^2 - \phi^2)A_\alpha A_\beta \ ] \quad ,$$

$$T_{4\alpha} = \frac{iN}{4\pi} \quad [\ (\vec{E} \times \vec{H})_\alpha + (1 + \vec{A}^2 - \phi^2)A_\alpha \phi] \quad ,$$

$$T_{44} = -\frac{N}{4\pi}\ [\tfrac{1}{2}(\vec{E}^2 + \vec{H}^2) + \tfrac{1}{2}(\vec{A}^2-\phi^2) + \tfrac{1}{4}(\vec{A}^2-\phi^2)^2 + K$$

$$+ \quad (1 + \vec{A}^2 - \phi^2)\phi^2] \ .$$

APPENDIX B: RELATION BETWEEN REST-MASS AND LAGRANGIAN

The energy of a system characterised by the potentials A_i, is (in the rest frame):

$$M = -\int T_{44} d^3x = \frac{N}{4\pi} \int \left[\frac{1}{4} F_{ik}{}^2 + \frac{1}{2} A_i{}^2 + \frac{1}{4} A_i{}^2 A_k{}^2 - F_{4k}{}^2 - (1 + A_k{}^2) A_4{}^2 + K\right] d^3x \quad .$$

From the field equations, (I-1), we obtain

$$\int A_i F_{ik,k}\, d^3x = -\int (1 + A_k{}^2) A_i{}^2 d^3x \quad \text{(no summation on } i\text{)}.$$

Since $A_i F_{ik,k} = (A_i F_{ik})_{,k} - A_{i,k} F_{ik}$ and

$$\int (A_i F_{ik})_{,k} d^3x = \int (A_i F_{i\alpha})_{,\alpha}\, d^3x + \int (A_i F_{i4})_{,4}\, d^3x$$

$$= \oint A_i F_{i\alpha} ds_\alpha + \int (A_i F_{i4})_{,4}\, d^3x \ ,$$

the above becomes

$$\oint A_i F_{i\alpha} ds_\alpha + \int [(A_i F_{i4})_{,4} - A_{i,k} F_{ik} + (1 + A_k{}^2) A_i{}^2] d^3x = 0 \quad .$$

Equivalently: 1.

$$\oint A_i F_{i\alpha} ds_\alpha + \int [(A_i F_{i4})_{,4} - A_{k,i} F_{ik} + F_{ik}^{\ 2} + (1 + A_k^{\ 2}) A_i^{\ 2}] d^3x = 0$$

With $i = 4$, this reduces the expression for the rest mass to

$$M = \frac{N}{4\pi} \int [\tfrac{1}{4} F_{ik}^{\ 2} + \tfrac{1}{2} A_i^{\ 2} + \tfrac{1}{4} A_i^{\ 2} A_k^{\ 2} + K] d^3x$$

$$- \frac{N}{4\pi} \int A_{\alpha,4} F_{4\alpha} d^3x + \frac{N}{4\pi} \oint A_4 F_{4\alpha} ds_\alpha ,$$

which is the relation (I-12).

---

1. For the static case, these integral relations follow from (I-10).

APPENDIX C: SOME UNPHYSICAL SOLUTIONS

The discussion of plane-wave solutions indicates the existence of analogous "standing wave" and "homogeneous" solutions corresponding to $\omega = 0$ and $\vec{k} = 0$ respectively. Many of these solutions are just limiting cases of the plane wave solutions. A large number of unphysical solutions may also be obtained by using the usual cylindrical coordinates (ρ, φ, z). For example, if $\vec{A} = 0$, the field equations reduce to:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} - \phi + \phi^3 = 0 .$$

For the case of cylindrical symmetry (when $\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial\varphi} = 0$) at least, phase space analysis of the type used by Finkelstein et al (15), indicates the existence of a countable (discrete) set of well behaved solutions asymptotic to zero as $\frac{e^{-\rho}}{\sqrt{\rho}}$, and a continuum of solutions asymptotic to ± 1 as

$$\frac{\sin(\sqrt{2}\,\rho + c)}{\sqrt{\rho}} .$$

Another example corresponds to $\vec{A} = A\vec{e}$ and $\phi = \gamma A$ where $\vec{e}$ is a unit vector fixed in space and γ a constant. For the steady state case $(\frac{\partial A}{\partial t} = 0)$, the field equations (I-1) become

$$\vec{e}\Delta A - \nabla(\vec{e}\cdot\nabla A) = (1 - \lambda^2 A^2)A\vec{e}$$

$$\Delta A = (1 - \lambda^2 A^2)A$$

with $\lambda^2 = \gamma^2 - 1$.

Choosing the z-axis along $\vec{e}$, this implies that $\vec{e}\cdot\nabla A = \frac{\partial A}{\partial z} = 0$ and $A = f(\rho,\varphi)$ where f is a solution of

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} = f - \lambda^2 f^3 ,$$

which has well behaved global solutions (at least for $\lambda^2 > 0$).

APPENDIX D: FIELD EQUATIONS IN SPHERICAL POLAR COORDINATES

The field equations (I-1), in spherical polar coordinates,

$$\vec{x} \equiv (r,\theta,\varphi) \qquad \vec{A} \equiv (A_r, A_\theta, A_\varphi)$$

take on the form:

$$\frac{1}{r^2}\frac{\partial^2 A_r}{d\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\varphi^2} - \frac{\partial^2 A_r}{\partial t^2} - \frac{\cot\theta}{r}\frac{\partial A_\theta}{\partial r}$$
$$- \frac{1}{r}\frac{\partial^2 A_\theta}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{\cot\theta}{r^2}A_\theta - \frac{1}{r\sin\theta}\frac{\partial^2 A_\varphi}{\partial r\partial\varphi} - \frac{1}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$
$$- \frac{\partial^2\phi}{\partial r\partial t} = (1 + A_r{}^2 + A_\theta{}^2 + A_\varphi{}^2 - \phi^2)A_r$$

$$\frac{1}{r}\frac{\partial^2 A_r}{\partial r\partial\theta} + \frac{\partial^2 A_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial A_\theta}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial\varphi^2} - \frac{\partial^2 A_\theta}{\partial t^2} - \frac{1}{r^2\sin\theta}\frac{\partial^2 A_\varphi}{\partial\theta\partial\varphi}$$
$$- \frac{\cot\theta\csc\theta}{r^2}\frac{\partial A_\varphi}{\partial\varphi} - \frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial t} = (1 + A_r{}^2 + A_\theta{}^2 + A_\varphi{}^2 - \phi^2)A_\theta$$

$$- \frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\varphi} - \frac{1}{r^2\sin\theta}\frac{\partial^2 A_\theta}{\partial\theta\partial\varphi} + \frac{\cot\theta\csc\theta}{r^2}\frac{\partial A_\theta}{\partial\varphi} + \frac{\partial^2 A_\varphi}{\partial r^2}$$
$$+ \frac{2}{r}\frac{\partial A_\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_\varphi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\varphi}{\partial\theta} - \frac{A_\varphi}{r^2\sin^2\theta} - \frac{\partial^2 A_\varphi}{\partial t^2} - \frac{1}{r\sin\theta}\frac{\partial^2\phi}{\partial\varphi\partial t}$$
$$= (1 + A_r{}^2 + A_\theta{}^2 + A_\varphi{}^2 - \phi^2)\ A_\varphi$$

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2 A_r}{\partial t\partial r} + \frac{2}{r}\frac{\partial A_r}{\partial t}$$

$$+ \frac{1}{r}\frac{\partial^2 A_\theta}{\partial t\partial\theta} + \frac{\cot\theta}{r}\frac{\partial A_\theta}{\partial t} + \frac{1}{r\sin\theta}\frac{\partial^2 A_\varphi}{\partial\varphi\partial t} = (1 + A_r{}^2 + A_\theta{}^2 + A_\varphi{}^2 - \phi^2)\phi$$

APPENDIX E: UNIQUENESS OF THE SPHERICALLY SYMMETRIC CAUCHY PROBLEM FOR $\Delta\phi = \phi - \phi^3$

Consider the Cauchy problem:

$$\Delta\phi = \phi - \phi^3 \quad \text{for} \quad r \leq R$$

with $\phi(R) = \pm 1$, $\left.\frac{\partial\phi}{\partial r}\right|_{r=R} = d$, $R > 0$ and d being constants.

Let $\phi(\vec{r}) = \pm 1 + \chi(\vec{r})$ be a solution of the problem. $\chi(\vec{r})$ then satisfies

$$\Delta\chi + 2\chi \pm \chi^2 + \chi^3 = 0$$

with $\chi = 0$, $\frac{\partial\chi}{\partial r} = d$ on $r = R$.

Near the surface $r = R$ (where $|\chi| << 1$)

$$\chi(\vec{r}) \simeq \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [A_{\ell m} j_\ell(x) + B_{\ell m} n_\ell(x)] Y_\ell^m (\theta,\varphi),$$

where $x = \sqrt{2}\, r$, j_ℓ and n_ℓ are spherical Bessel and Neumann functions of order ℓ respectively (Ref. (17), p. 539), and $A_{\ell m}$, $B_{\ell m}$ are constants.

At $r = R$, $\chi(R,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [A_{\ell m} j_\ell(X) + B_{\ell m} n_\ell(X)] Y_\ell^m(\theta,\varphi) = 0$

hence $\quad A_{\ell m} j_\ell(X) + B_{\ell m} n_\ell(X) = 0$ for all ℓ and m,
where $X = \sqrt{2}\, R$.

Similarly, from $\left.\frac{\partial\phi}{\partial r}\right|_{r=R} = d$, we get

$$A_{\ell m} j'_\ell\,(X) + B_{\ell m} n'_\ell\,(X) = 0 \quad \text{for all } \ell > 0$$

and $|m| \leq \ell$. (The prime denotes differentiation with respect to the argument). If X is a zero of one of j_ℓ, n_ℓ or their first derivatives, then these conditions imply that

$$A_{\ell m} = B_{\ell m} = 0 \quad \text{for } \ell > 0, |m| \leq \ell \ .$$

Otherwise $\quad B_{\ell m} = -\dfrac{j_\ell(X)}{n_\ell(X)} A_{\ell m}$, hence

$$\frac{A_{\ell m}}{n_\ell(X)} W\,[n_\ell(x), j_\ell(x)]_{x=X} = 0,$$

W being the Wronskian of n_ℓ and j_ℓ, that is

$$W\,[n_\ell, j_\ell]_{x=X} = \frac{1}{2R^2} \qquad \text{(Ref. (17), p. 541).}$$

It then follows that $A_{\ell m} = B_{\ell m} = 0$ for all $\ell > 0$, $|m| \leq \ell$, so that $\phi(\vec{r})$ is spherically symmetric in the neighbourhood of $r = R$. However $f(r) = r\,\phi(r)$ satisfies (in the limit as $\delta r, \delta\mu \to 0$) the difference equation (V-13), from which it is obvious that since the solution is spherically symmetric at $r = R$ and $r = R - \delta r$ (for arbitrarily small $\delta r > 0$), then it is such for $r = R - 2\delta r$, hence by induction for all $r \leq R$.

APPENDIX F: EXISTENCE OF SOLUTIONS FOR (IV-4)

For the system of linear, inhomogeneous differential equations (IV-4) assume that the solutions for $i = 1, 2, \ldots, \ell$ are polynomials of degree $\leq i$ for each i. Consider now the equation for $i = \ell + 2$:

$$P[\alpha_{\ell+2}(\mu)]^{1.} + (\ell+3)(\ell+2)\alpha_{\ell+2}(\mu) = \beta_{\ell+2}(\mu) \qquad \text{(F-1)}$$

where

$$\beta_{\ell+2}(\mu) = \alpha_\ell - \sum_{m=0}^{\ell} \sum_{n=0}^{m} \alpha_{\ell-m}\alpha_{m-n}\alpha_n =$$

$$\sum_{j=0}^{\ell} b_{\ell j} P_j(\mu) \qquad \text{(say).} \qquad \text{(F-2)}$$

Assume $\alpha_{\ell+2}(\mu) = \sum_{j=0}^{\infty} a_{\ell j} P_j(\mu) = \gamma_{\ell+2}(\mu)$. Substituting into (F-1) we find that,

$$\sum_{j=0}^{\ell} [-j(j+1)a_{\ell j} + (\ell+3)(\ell+2)a_{\ell j} - b_{\ell j}]\, P_j(\mu)$$

$$+ \sum_{j=\ell+1}^{\infty} [-j(j+1)+(\ell+3)(\ell+2)]\, a_{\ell j} P_j(\mu) = 0$$

from which it follows that (F-1) has the solution

---

1. P[ ] is defined on p. 27.

$$a_{\ell j} = \frac{b_{\ell j}}{(\ell+3)(\ell+2)-j(j+1)} \qquad \text{for } j = 0, 1, 2, .., \ell$$

$$= 0 \qquad \text{for } j > \ell.$$

Furthermore, this solution is a polynomial of degree ℓ, hence is analytic everywhere in $|\mu| \leq 1$.

For $\ell = 0$, (IV-4) has the solution

$$\alpha_2(\mu) = C_2 P_2(\mu) + \frac{1}{6}(C_o - C_o{}^3)$$

which is of the form (F-2) as assumed (with $b_{20} = \frac{1}{6}(C_o - C_o{}^3)$, $b_{21} = 0$ and $b_{22} = C_2$). Thus, the existence of well behaved solutions for all α_ℓ follows by induction.

APPENDIX G: PARITY OF SOLUTIONS OF (IV-4)

As shown in Chapter IV and Appendix F, the solutions of (IV-4) may be written as

$$\alpha_k(\mu) = C_k P_k(\mu) + \sum_{\ell=0}^{k-2} \frac{b_{k\ell} P_\ell(\mu)}{(k+1)k-(\ell+1)\ell}$$

where $b_{k\ell}$ are obtainable from

$$\sum_{\ell=0}^{k-2} b_{k\ell} P_\ell(\mu) = \alpha_{k-2}(\mu) - \sum_{m=0}^{k-2} \alpha_{k-m-2}(\mu)\alpha_{m-n}(\mu)\alpha_n(\mu) \ . \qquad \text{(G-1)}$$

Assume that

$$\alpha_\ell(-\mu) = (-1)^\ell \alpha_\ell(\mu) \qquad \text{for } \ell = 0,1,\ldots,k-2 \qquad \text{(G-2)}$$

and consider

$$\alpha_k(-\mu) = (-1)^k C_k P_k(\mu) = \sum_{\ell=0}^{k-2} \frac{(-1)^\ell b_{k\ell} P_\ell(\mu)}{(k+1)k-(\ell+1)\ell} \ . \qquad \text{(G-3)}$$

Replacing μ by $-\mu$ in (G-1), we obtain, using (G-2),

$$\sum_{\ell=0}^{k-2} (-1)^\ell b_{k\ell} P_\ell(\mu) = (-1)^{k-2}\alpha_{k-2}(\mu) - \sum_{m=0}^{k-2} (-1)^{k-m-2}\alpha_{k-m-2}(\mu)(-1)^{m-n}$$

$$\alpha_{m-n}(\mu)(-1)^n\alpha_n(\mu),$$

$$= \quad (-1)^{k-2} \sum_{\ell=0}^{k-2} b_{k\ell} P_\ell(\mu).$$

Thus $$\sum_{\ell=0}^{k-2} \frac{(-1)^{\ell} b_{k\ell} P_{\ell}(\mu)}{(k+1)k-(\ell+1)\ell} = (-1)^k \sum_{\ell=0}^{k-2} \frac{b_{k\ell} P_{\ell}(\mu)}{(k+1)k-(\ell+1)\ell} .$$

Substituting into (G-3) we get $\alpha_k(-\mu) = (-1)^k \alpha_k(\mu)$. Since (G-2) holds for $\ell = 0,1,2,$ then by induction it holds for all ℓ.

APPENDIX H. RADIUS OF CONVERGENCE

Consider the radius of convergence of the series

$$f(r) = \sum_{\ell=0}^{\infty} \alpha_\ell \, r^{2\ell+1} \tag{H-1}$$

where

$$(2\ell+3)(2\ell+2)\,\alpha_{\ell+1} = \alpha_\ell - \sum_{m=0}^{\ell} \sum_{n=0}^{m} \alpha_{\ell-m}\alpha_{m-n}\alpha_n . \tag{H-2}$$

Assume that $|\alpha_i| \leq M^i$ for $i \leq \ell$, (H-3)

where M is the largest of $\frac{1}{6}|\alpha_o - \alpha_o^3|$, $|\alpha_o|$ and 1.

Then, from (H-2)

$$(2\ell+3)(2\ell+2)\,|\alpha_{\ell+1}| \leq |\alpha_\ell| + \sum_{m=0}^{\ell} \sum_{n=0}^{m} |\alpha_{\ell-m}|\,|\alpha_{m-n}|\,|\alpha_n|$$

$$\leq M^\ell + \sum_{m=0}^{\ell} \sum_{n=0}^{m} M^{\ell-m} M^{m-n} M^n \quad \text{by (H-3).}$$

Thus

$$(2\ell+3)(2\ell+2)\,|\alpha_{\ell+1}| \leq M^\ell + \sum_{m=0}^{\ell} \sum_{n=0}^{m} M^\ell = M^\ell \left[1 + \frac{(\ell+1)(\ell+2)}{2}\right]$$

or

$$|\alpha_{\ell+1}| \leq \frac{2 + (\ell+1)(\ell+2)}{2(2\ell+3)(2\ell+2)} M^\ell \leq M^{\ell+1} .$$

Since (H-3) is true for $\ell = 1$ $(\alpha_1 = \frac{1}{6}(\alpha_0 - \alpha_0{}^3))$, then by induction (H-3) holds for all ℓ.

Thus we have

$$\sum_{\ell=0}^{\infty} \alpha_\ell \, r^{2\ell+1} \leq \sum_{\ell=0}^{\infty} M^\ell \, r^{2\ell+1} \quad .$$

But, by the ratio test, the latter series converges uniformly to an analytic function for

$$r < M^{-\frac{1}{2}} \quad ,$$

hence the same applies to (H-1).

APPENDIX J. SERIES SOLUTION OF (IV-6)

In the table below we list some examples of the solutions of (IV-6) as evaluated from the series (IV-5) for various $\phi(0)$. K is the number of terms of the series required for the tabulated results to be exact to (at least) four decimal places. The last line, for each $\phi(0)$ corresponds to that value of r for which the series first appears to diverge (then $\phi_{300} = \sum_{\ell=0}^{300} \alpha_\ell r^{2\ell+1}$).

$\phi(0) = .5$

r	ϕ	K
0.200	0.50250	2
0.400	0.51002	2
0.600	0.52259	3
0.800	0.54028	3
1.000	0.56314	4
1.200	0.59119	4
1.400	0.62440	4
1.600	0.66262	5
1.800	0.70551	5
2.000	0.75256	6
2.200	0.80287	7
2.400	0.85521	7
2.600	0.90781	9
2.800	0.95877	10
3.000	1.00565	10
3.200	1.04575	10
3.400	1.07846	15
3.600	1.10047	20
3.800	1.11187	23
4.000	1.11571	23
4.200	1.14296	23
4.400	1.09115	117
4.6	$\phi_{300}=1.204\ 10^6$	$\alpha_{301}=-6.313\ 10^5$

$\phi(0) = 1.25$

r	ϕ	K
0.200	1.24535	2
0.400	1.23179	3
0.600	1.21047	4
0.800	1.18311	4
1.000	1.15168	5
1.200	1.11833	6
1.400	1.08495	8
1.600	1.05323	10
1.800	1.02445	13
2.000	0.99961	17
2.200	0.97937	26
2.400	0.96393	48
2.600	0.95326	275
2.8	$\phi_{300}=4.485\ 10^{14}$	$\alpha_{301}=8.484\ 10^{14}$

$\phi(0) = 2$

r	ϕ	K
0.200	1.96086	3
0.400	1.85300	4
0.600	1.70002	6
0.800	1.52888	10
1.000	1.36137	16
1.200	1.21093	35
1.4	$\phi_{300}=3.579\ 10^{2}$	$\alpha_{301}=7.067\ 10^{2}$

$\phi(0) = 5$

r	ϕ	K
0.100	4.80714	4
0.200	4.30317	6
0.300	3.64990	10
0.400	2.99075	21
0.500	2.40473	136
.6	-	$\alpha_{239}=1.311\ 10^{30}$

$\phi(0) = 10$

r	ϕ	K
0.100	8.56449	7
0.200	5.86661	23
.3	-	α_{224}=-1.350 10^{30}

APPENDIX K. VARIATIONAL APPROXIMATION TO POSSIBLE ODD-PARITY NEUTRAL PARTICLE EIGENSOLUTIONS USING THE COMPARISON FUNCTION $\phi_A = R(r)\ P_1(\mu)$

Substitution of the comparison function $\phi_A = R(r)P_1(\mu)$ into the Lagrangian (VI-2), yields

$$M = \frac{1}{6} \int_0^\infty [R'^2 + R^2 (1 + \frac{2}{r^2}) - \frac{3}{10} R^4]\ r^2\ dr \qquad \text{(K-1)}$$

The corresponding Euler-Lagrange equation is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - (1 + \frac{2}{r^2})R + \frac{3}{5} R^3 = 0\ . \qquad \text{(K-2)}$$

In analogy to (IV-6), this equation has a discrete set of "orthogonal" [1] eigensolutions asymptotic to zero as $A_i \frac{e^{-r}}{r} (1 + \frac{1}{r}) = A_i\ K_1(r)$, [2] which, presumably, yield upper bounds to the masses of the particles represented by odd-parity eigensolutions of (III-4).

---

1 Orthogonal in the sense of (VI-1).

2 The first three values of the asymptotic amplitude A_i are listed in table (VI-B), while the first three eigensolutions are plotted in figure 6.6.

From (K-2) it follows that the solutions satisfy

$$\int_0^\infty \left[\left(\frac{dR}{dr}\right)^2 + \left(1 + \frac{2}{r^2}\right)R^2 - \frac{3}{5} R^4\right]r^2 \, dr = 0. \tag{K-3}$$

which reduces (K-1) to

$$M = \frac{1}{20} \int_0^\infty R^4 \, r^2 \, dr \, .$$

Note that the eigensolutions of (K-2) immediately satisfy the integral relations (V-4) and (V-6), since the latter are satisfied by all comparison functions for which L has been extremised with respect to amplitude and radial scale parameters.[1]

Near the origin, the solutions of (K-2) may be written[2]

$$R = \sum_{\ell=1}^{\infty} a_\ell \, r^{2\ell-1} ;$$

where a_1 is arbitrary, while

$$[(2\ell+3)(2\ell+2) - 2] \, a_{\ell+1} = a_\ell - \frac{3}{5} \sum_{m=0}^{\ell} \sum_{n=0}^{m} a_{\ell-m} \, a_{m-n} \, a_n$$

for $\ell = 1, 2, \ldots$.

---

1 Thus (K-3) is obtainable from (V-6).

2 This series expansion of R in the vicinity of $r = 0$ is obtainable in the same fashion as (IV-5).

The first three eigensolutions of (K-2), which are plotted in figure 6.6, were obtained numerically by solving the finite difference approximation

$$\frac{f(r+\delta) + f(r-\delta) - 2f(r)}{\delta^2} = (1 + \frac{2}{r^2})f(r) - \frac{3}{5}\frac{f^3(r)}{r^2}$$

to (K-2) ($f = rR$ and δ is the mesh size), using the method described in Chapter V.

The integral expressions which are given in table VIII B were obtained numerically from

$$E_r = \frac{2}{5}\int_0^\infty [\frac{1}{5} r^5 RR'^3 - \frac{1}{3} r^4 R'^2 + \frac{3}{5} r^4 R^2 R'^2] \, dr,$$

$$E_x = \frac{1}{5}\int_0^\infty [\frac{1}{3} r^2 R'^2 - \frac{2}{3} rRR' + \frac{1}{3} R^2] \, dr \quad ,$$

and

$$E_z = \frac{1}{5}\int_0^\infty [r^2 R'^2 + \frac{4}{3} rRR' + \frac{8}{3} R^2] \, dr$$

where $R' \equiv \frac{dR}{dr}$.

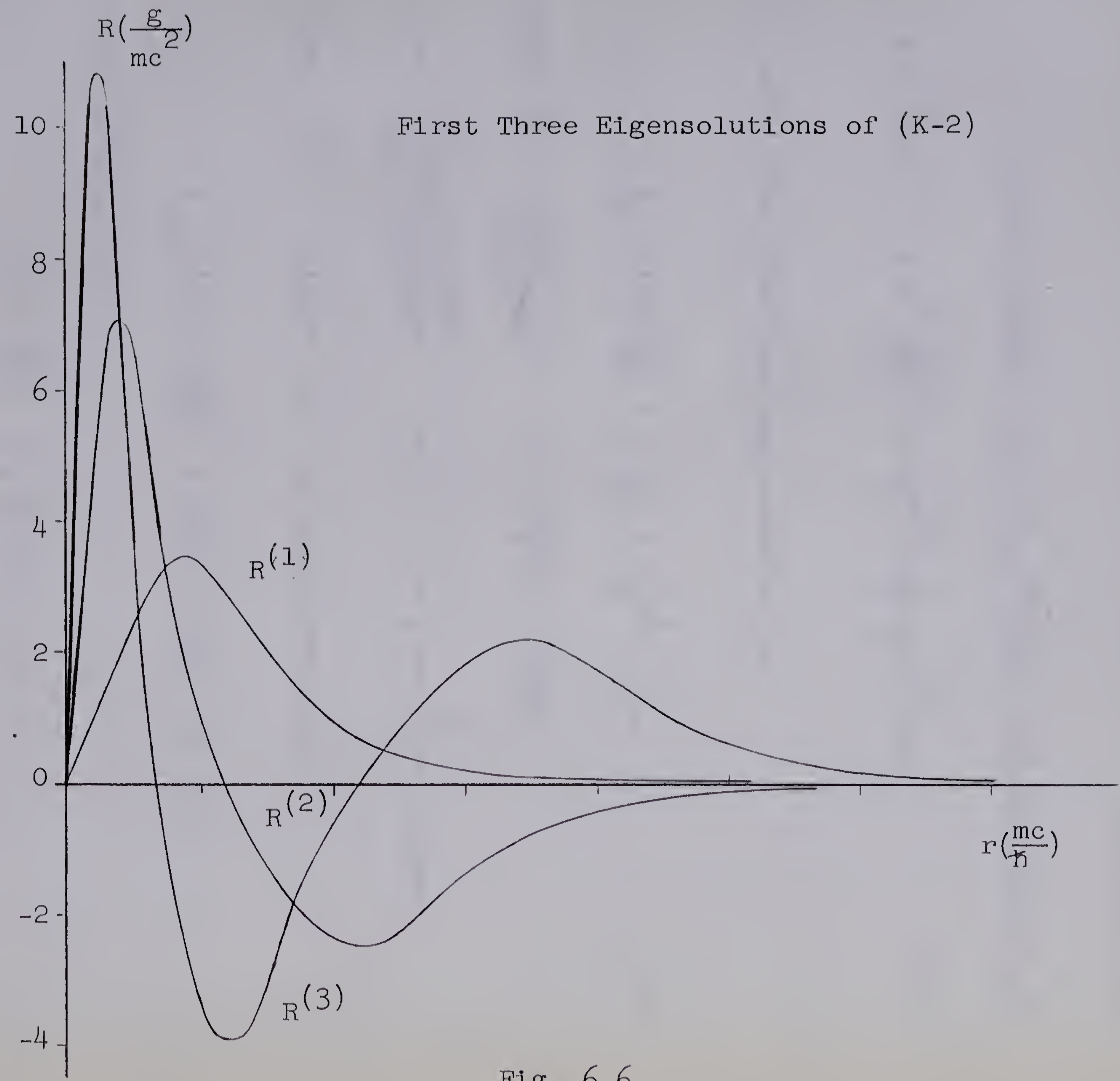

R(g/mc²)
10
8
6
4
2
0
-2
-4
First Three Eigensolutions of (K-2)
R(1)
R(2)
R(3)
r(mc/ħ)

Fig. 6.6

APPENDIX L. VARIATIONAL APPROXIMATIONS TO ODD-PARITY NEUTRAL PARTICLE EIGENSOLUTIONS USING THE TRIAL FUNCTION $\phi_B = r^n e^{-\alpha r} Y(\mu)$

The function $\phi_B = r^n e^{-\alpha r} Y(\mu)$ reduces (VI-2) to[1]

$$M = \frac{C_o}{4} \int_{-1}^{1} \left[(1-\mu^2)\left(\frac{dY}{d\mu}\right)^2 + C_2 Y^2 - \frac{1}{2} C_4 Y^4 \right] d\mu \quad , \tag{L-1}$$

for which the Euler-Lagrange equation is

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dY}{d\mu}\right] - C_2 Y + C_4 Y^3 = 0 \ , \tag{L-2}$$

where $C_o = \dfrac{(2n)!}{(2\alpha)^{2n+1}}$, $C_2 = \dfrac{n+1}{2}\left(1 + \dfrac{2n+1}{\alpha^2}\right)$,

$C_4 = \dfrac{(4n+2)! \ (2\alpha)^{2n+1}}{(2n)! \ (4\alpha)^{4n+3}}$ and n is a positive integer.

From (L-2) it follows that the solutions satisfy

$$\int_{-1}^{1} \left[(1-\mu^2)\left(\frac{dY}{d\mu}\right)^2 + C_2 Y^2 - C_4 Y^4\right] d\mu = 0$$

which reduces (L-1) to

$$M = \frac{C_o C_4}{8} \int_{-1}^{1} Y^4 \, d\mu \quad .$$

---

1 Recall that $M = -L$ for neutral particles.

Setting $v = 1 - \mu$, (L-2) becomes

$$\frac{d}{dv}[(2v - v^2)\frac{dY}{dv}] - C_2 Y + C_4 Y^3 = 0 \qquad \text{(L-3)}$$

with $Y(v = 1) = 0$ for an odd parity solution.

Near $v = 0$ $\quad Y = \sum_{\ell=0}^{\infty} b_\ell \, v^\ell$, where $b_o = Y(\mu = 1)$

is arbitrary, while

$$2(\ell+1)^2 b_{\ell+1} = [(\ell+1)\ell + C_2]\, b_\ell - C_4 \sum_{m=0}^{\ell} \sum_{n=0}^{m} b_{\ell-m}\, b_{m-n}\, b_n$$

for $\ell = 1, 2, \ldots$.

The finite difference approximation to (L-3) is

$$(2v-v^2)\left[\frac{Y(v+\delta)+Y(v-\delta)-2Y(v)}{\delta^2}\right] + 2(1-v)\left[\frac{Y(v+\delta)-Y(v-\delta)}{2\delta}\right] - C_2 Y(v) + C_4 Y^3(v)$$

$$= 0$$

Numerical solution of this equation, using the iteration technique described in Chapter V indicates the existence of a discrete set of orthogonal (in the sense of VI-1) odd-parity eigensolutions of (L-2) for each value of n and α, corresponding to a discrete set of $Y(\mu=1)$. The results are given in table VI B and in figures 6.7 to 6.9 below.

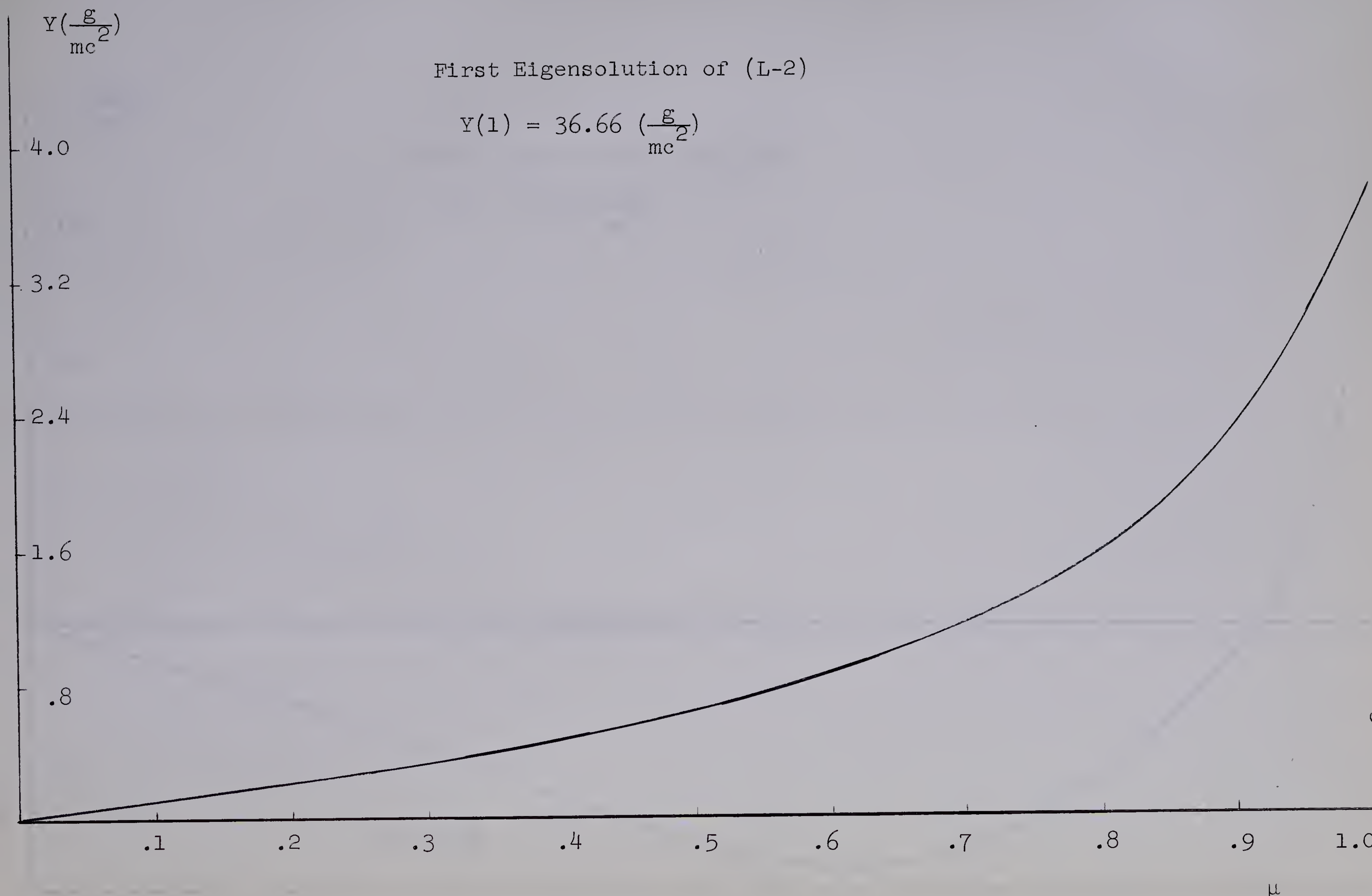
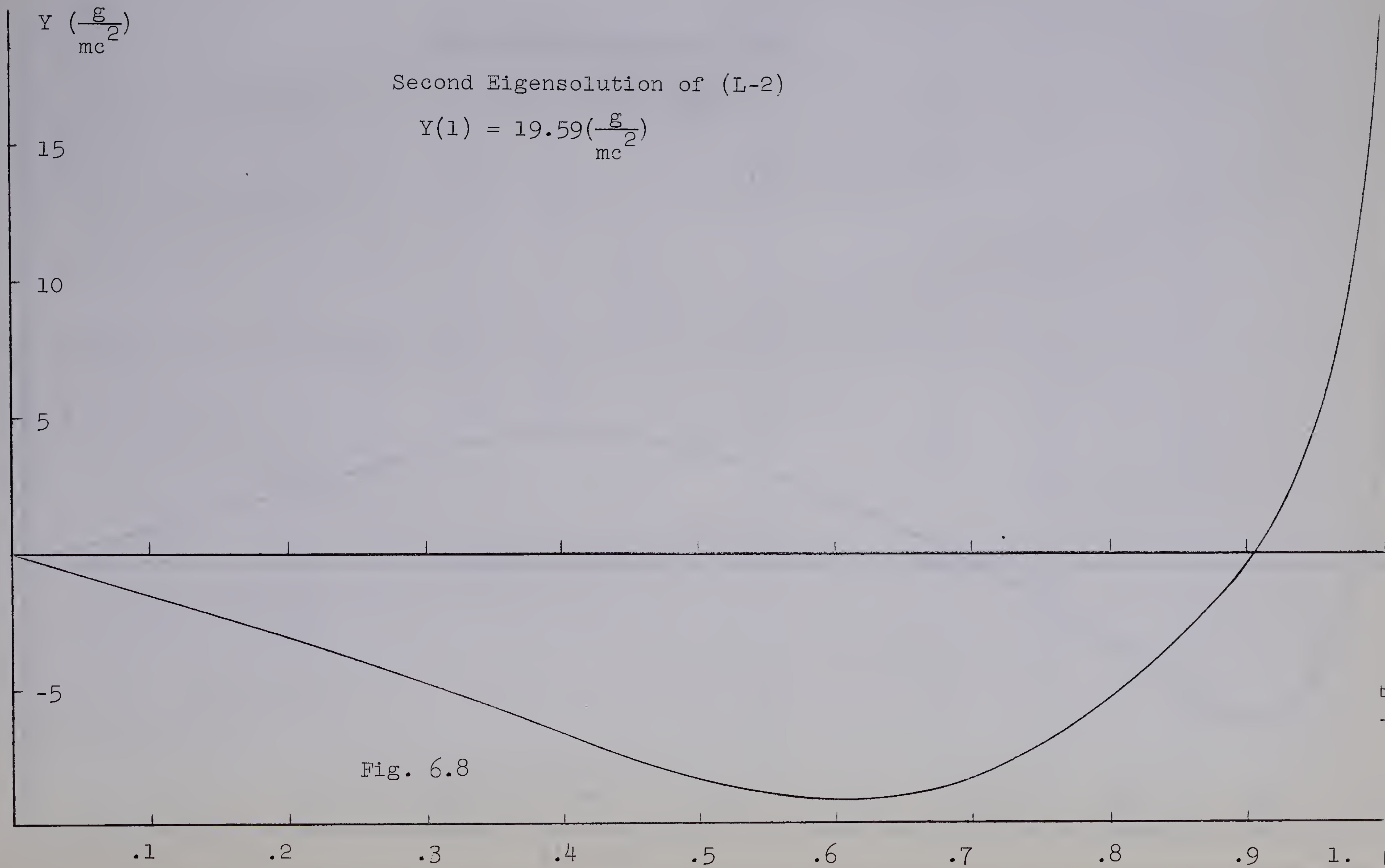
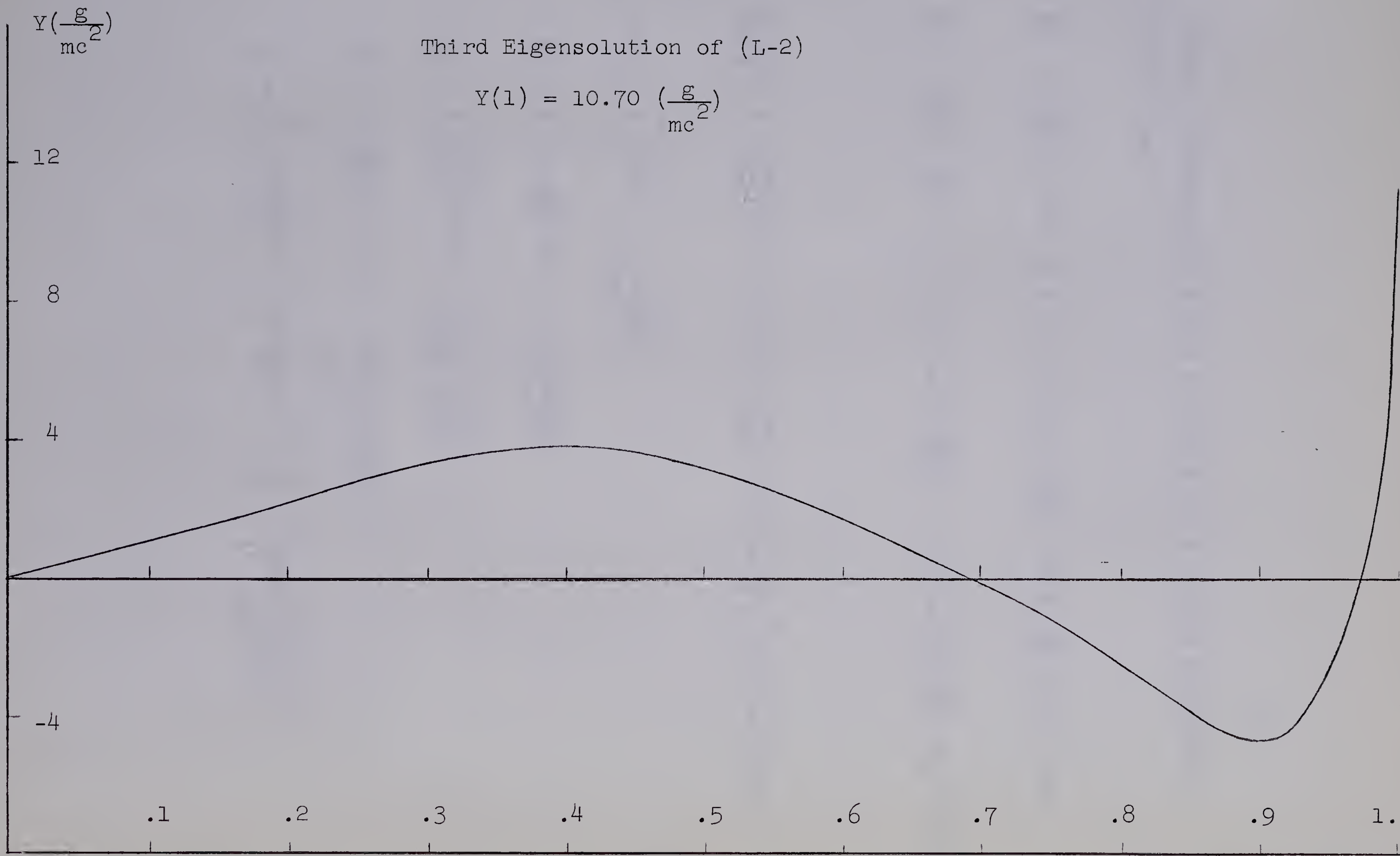

Y($\frac{g}{mc^2}$)
First Eigensolution of (L-2)
Y(1) = 36.66 ($\frac{g}{mc^2}$)
4.0
3.2
2.4
1.6
.8
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
μ

Fig. 6.7



$Y\ (\frac{g}{mc^2})$
Second Eigensolution of (L-2)
$Y(1) = 19.59(\frac{g}{mc^2})$
15
10
5
-5
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.
μ

Fig. 6.8



Third Eigensolution of (L-2)
Y(1) = 10.70 (g/mc2)
Y(g/mc2)
12
8
4
-4
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.
μ

Fig. 6.9

The integral expressions given in table VIII B were evaluated numerically from

$$E_x = \frac{1}{4\pi} \int \left(\frac{\partial\phi}{\partial x}\right)^2 d^3x = \frac{1}{2}\int_0^1 (1-\mu^2)[c_o\mu^2(\frac{dY}{d\mu})^2 - 2\mu\, g_o Y\frac{dY}{d\mu} + g_2 Y^2]\, d\mu,$$

$$E_z = \frac{1}{4\pi} \int \left(\frac{\partial\phi}{\partial z}\right)^2 d^3x = \int_0^1 [c_o(1-\mu^2)^2(\frac{dY}{d\mu})^2 + 2\mu(1-\mu^2)g_o Y\frac{dY}{d\mu} + g_2\mu^2 Y^2]\, d\mu,$$

and

$$E_r = \frac{1}{4\pi}\frac{2}{5} \int [\ \phi(\frac{\partial\phi}{\partial r})^3 r^3 + r^2(3\phi^2-1)(\frac{\partial\phi}{\partial r})^2\ ] d^3x = \frac{2}{5}\int_0^1 (s_4 Y^4 - s_2 Y^2)\, d\mu,$$

where

$$c_o = \int_0^\infty R^2 dr = \frac{(2n)!}{(2\alpha)^{2n+1}} ,$$

$$g_o = \int_0^\infty r\, R\frac{dR}{dr}\, dr = -\frac{(2n)!}{2(2\alpha)^{2n+1}} ,$$

$$g_2 = \int_0^\infty (\frac{dR}{dr})^2 r^2 dr = \frac{(2n)!\ (n+1)}{2(2\alpha)^{2n+1}} ,$$

$$s_2 = \int_0^\infty (\frac{dR}{dr})^2\ r^4\ dr = \frac{(n+2)!\ (n+6)}{2(2\alpha)^{2n+3}} ,$$

$$s_4 = \int_0^\infty [R(\frac{dR}{dr})^3 r^5 + 3R^2(\frac{dR}{dr})^2 r^4]\ dr = \frac{(4n+2)!(n+21)}{16\ (4\alpha)^{4n+3}} .$$

APPENDIX M. VARIATIONAL APPROXIMATIONS TO ODD PARITY NEUTRAL PARTICLE EIGENSOLUTIONS USING THE TRIAL FUNCTION $\phi_C = R(r)P_1(\mu)+Q(r)P_3(\mu)$

The comparison function $\phi_C = R(r)P_1(\mu) + Q(r)P_3(\mu)$ reduces the Lagrangian (VI-1a) to

$$M = \frac{1}{2}\int_0^\infty r^2[\tfrac{1}{3}(\tfrac{dR}{dr})^2 + \tfrac{1}{7}(\tfrac{dQ}{dr})^2 + \tfrac{1}{3}(1+\tfrac{2}{r^2})R^2 + \tfrac{1}{7}(1+\tfrac{12}{r^2})Q^2 - \tfrac{1}{5}(\tfrac{1}{2}R^4 + \tfrac{4}{7}R^3Q + \tfrac{23}{21}R^2Q^2 + \tfrac{24}{77}RQ^3 + \tfrac{241}{2002}Q^4)]dr. \qquad (M\text{-}1)$$

Extremisation of (M-1) with respect to the functions R and Q yields the set of two ordinary, non-linear, coupled equations

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr}) - (1+\frac{2}{r^2})R + \frac{3}{5}(R^3 + \frac{6}{7}R^2Q + \frac{23}{21}RQ^2 + \frac{12}{77}Q^3) = 0 \qquad (M\text{-}2)$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dQ}{dr}) - (1+\frac{12}{r^2})Q + \frac{1}{5}(2R^3 + \frac{23}{3}R^2Q + \frac{36}{11}RQ^2 + \frac{241}{143}Q^3) = 0 \qquad (M\text{-}3)$$

Multiplying (M-2) and (M-3) by R and Q respectively, and integrating each over all space, implies

$$\int_0^\infty r^2[(\frac{dR}{dr})^2 + (1+\frac{2}{r^2})R^2 - \frac{3}{5}(R^4 + \frac{6}{7}R^3Q + \frac{23}{21}R^2Q^2 + \frac{12}{77}Q^3R)]dr = 0,$$

$$\int_0^\infty r^2[(\frac{dQ}{dr})^2 + (1+\frac{12}{r^2})Q^2 - \frac{1}{5}(2R^3Q + \frac{23}{3}R^2Q^2 + \frac{36}{11}RQ^3 + \frac{241}{143}Q^4)]dr = 0,$$

whereupon (M-1) reduces to

$$M = \frac{1}{10}\int_0^\infty r^2(\frac{1}{2}R^4 + \frac{4}{7}R^3Q + \frac{23}{21}R^2Q^2 + \frac{24}{77}RQ^3 + \frac{241}{2002}Q^4)dr = 0$$

which is just $M = \frac{1}{16\pi}\int \phi_C^4\, d^3x$.

If R and Q satisfy (M-2) and (M-3) then it follows, as in the case of Φ_A (Appendix K), that Φ_C satisfies the integral relations (V-4) and (V-6). Note, however, that Φ_C does not satisfy (V-7) since in extremising (M-1) no extremisation with respect to rectangular cartesian scale parameters takes place.[1]

For large r, solutions of (M-2) and (M-3) decouple and take the form

$$R(r) = A_i k_1(r),$$

$$Q(r) = B_i k_3(r).$$ [2]

Near the origin,

$$R(r) = a(r + \frac{1}{10}r^3 + \ldots.)$$ [3]

and

$$Q(r) = br^3 + \frac{1}{18}(b - \frac{2}{5}a^3)r^5 + \ldots.$$

Using this information, solutions of (M-2) and (M-3) which are asymptotic to zero, may be obtained numerically by

---

1 Values of these integrals, for the first few eigensolutions Φ_C, are listed in table VIII B

2 $k_i(r)$ are defined on p. 44.

3 Note that $\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{r} - \frac{\sin\theta}{r}\frac{\partial\Phi}{\partial\mu}\hat{\theta}$ exists everywhere, in particular at the origin where $\Phi = a(\mu\hat{r} - \sin\theta\,\hat{\theta}) = a\,\hat{z}$.

replacing the derivatives with central difference approximations and solving the resulting difference equations as described in Chapter V. The first few eigensolutions, obtained in this way, are plotted in figures 6.10 to 6.13.

The integral expressions of table VIII B take on the form

$$E_r = \frac{2}{5}(E_{r1} - E_{r2} + E_{r3}) \; ,$$

where

$$E_{r1} = \frac{1}{5}\int_0^\infty r^5 \, [RR'^3 + \frac{2}{7}R'^2(3RQ' + R'Q) + \frac{23}{21}R'Q'(RQ' + R'Q)$$

$$\frac{12}{77}Q'^2(RQ' + 3R'Q) + \frac{241}{1001}\, QQ'^3]dr \; ,$$[1]

$$E_{r2} = \int_0^\infty r^4[\frac{1}{3}\, R'^2 + \frac{1}{7}\, Q'^2]dr \; ,$$

$$E_{r3} = \frac{3}{5}\int_0^\infty r^4[R^2R'^2 + \frac{4}{7}RR'(RQ' + R'Q) + \frac{23}{63}(R^2Q'^2 + 4RR'QQ' + R'^2Q^2)$$

$$+ \frac{24}{77}QQ'(RQ' + R'Q) + \frac{241}{1001}Q^2Q'^2]dr \; ,$$

$$E_z = \frac{1}{5}\int_0^\infty [r^2(R'^2 + \frac{4}{7}\, R'Q' + \frac{23}{63}Q'^2) + 4r(\frac{1}{3}RR' - \frac{1}{7}RQ' + \frac{4}{7}R'Q + \frac{2}{21}QQ')$$

$$+ 8(\frac{1}{3}R^2 - \frac{2}{7}RQ + \frac{4}{7}Q^2)]dr \; ,$$

and

$$E_x = \frac{1}{5}\int_0^\infty [r^2(\frac{1}{3}R'^2 - \frac{2}{7}R'Q' + \frac{11}{63}Q'^2) - 2r(\frac{1}{3}RR' - \frac{1}{7}RQ' + \frac{4}{7}R'Q + \frac{2}{21}QQ')$$

$$+ \frac{1}{3}R^2 + \frac{8}{7}RQ + 2Q^2]dr.$$

---

1 $R' = \frac{dR}{dr}$, etc.

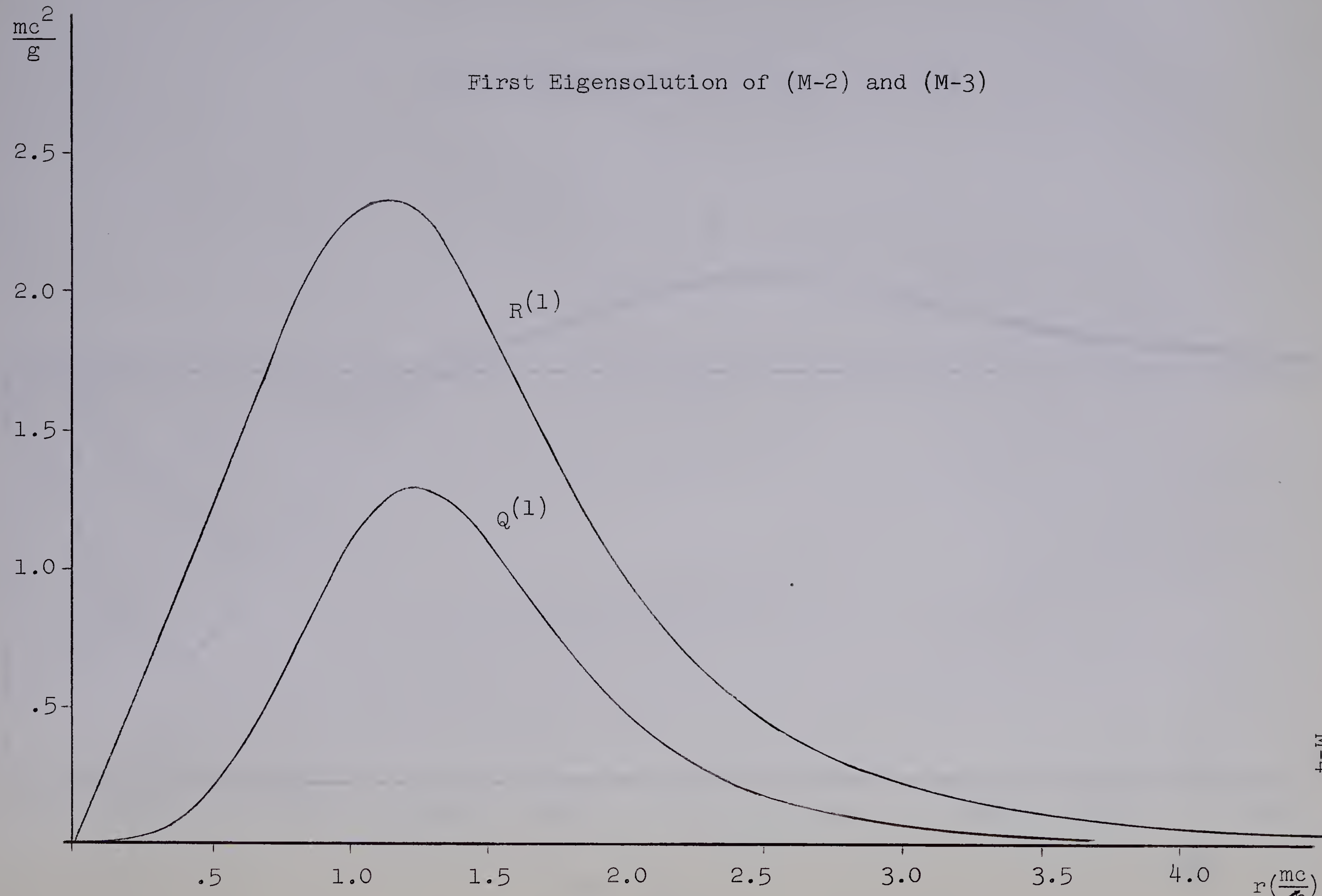

First Eigensolution of (M-2) and (M-3)
$\frac{mc^2}{g}$
2.5
2.0
1.5
1.0
.5
$R^{(1)}$
$Q^{(1)}$
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
$r(\frac{mc}{\hbar})$

Fig. 6.10

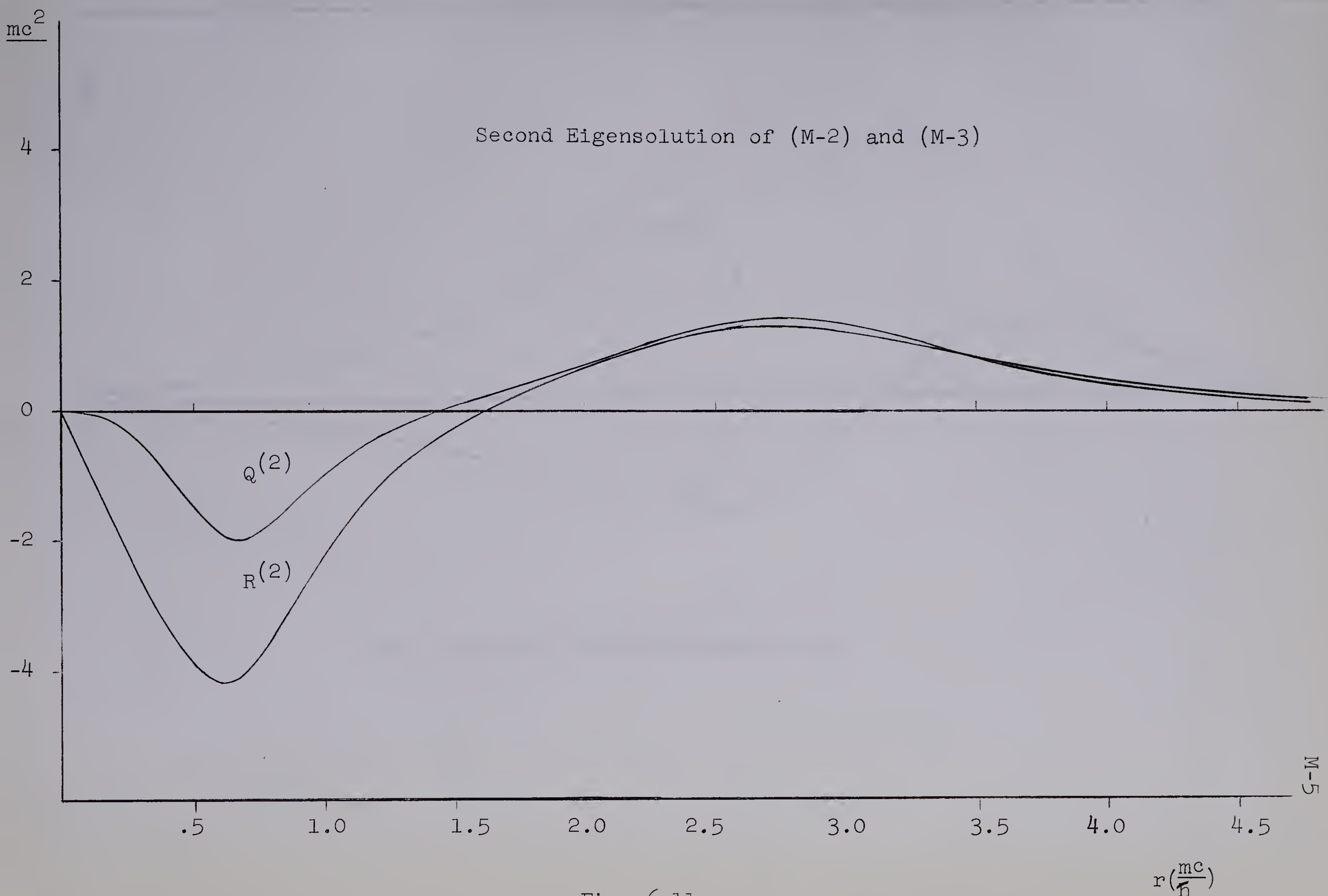
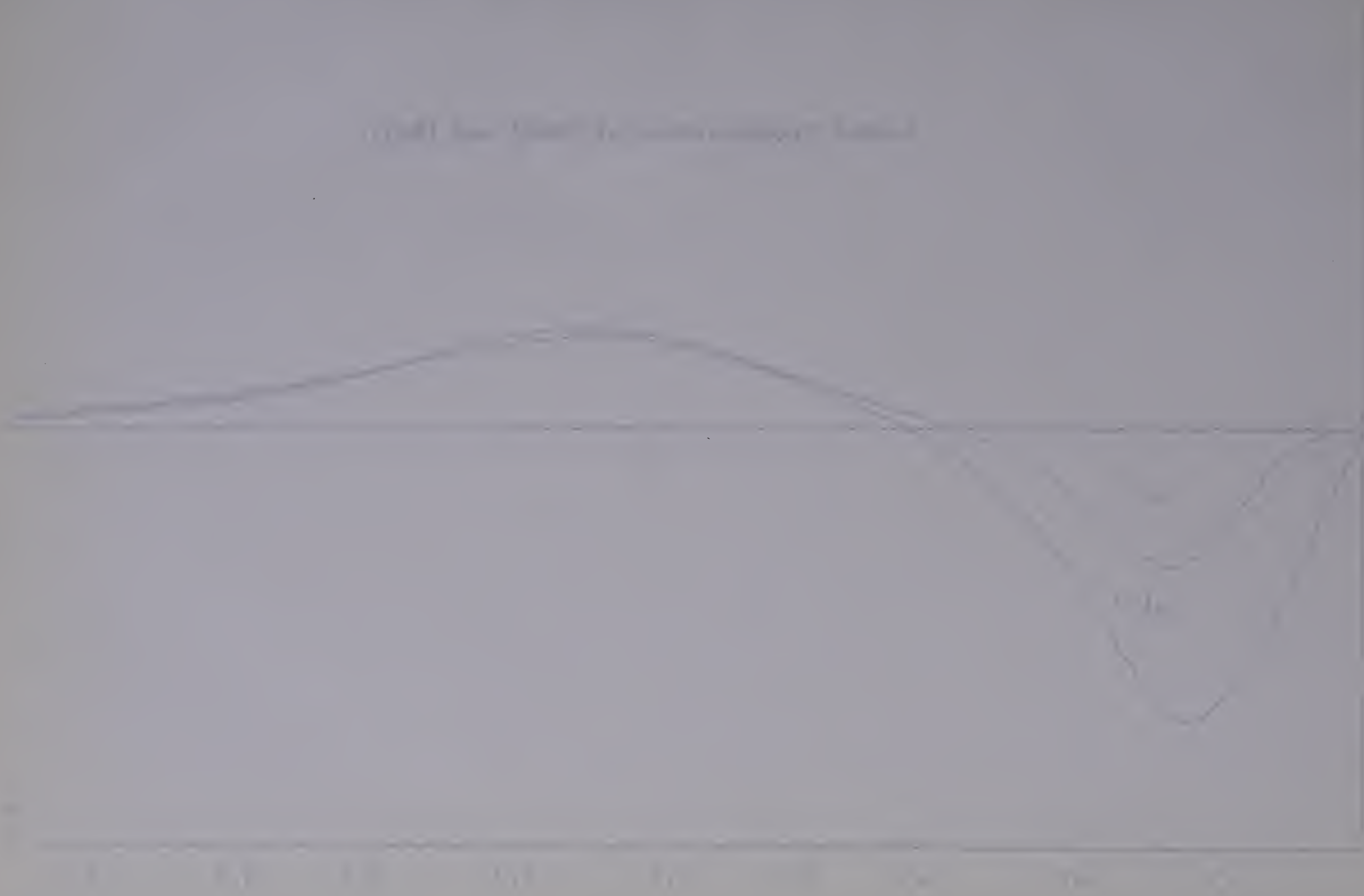
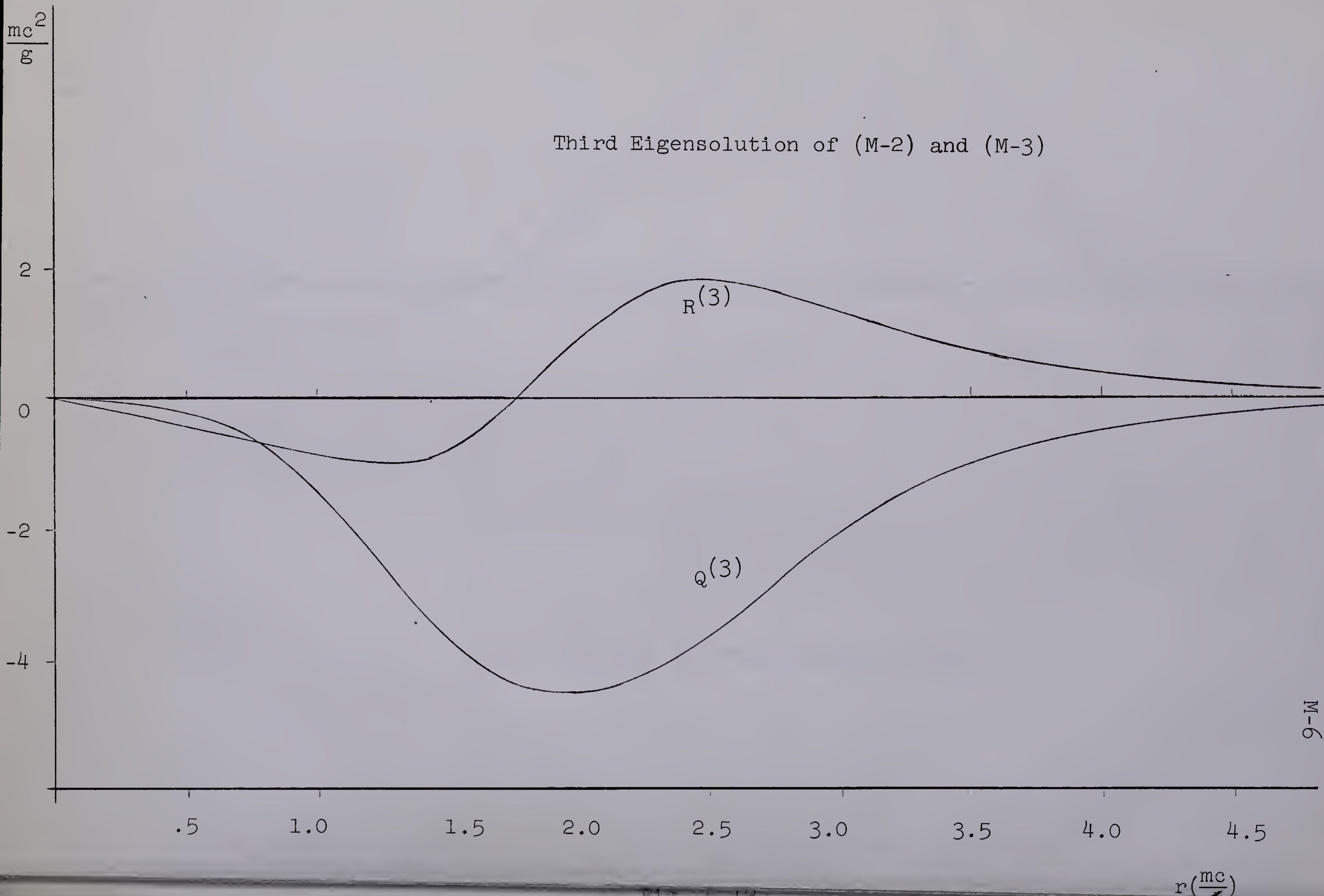

Second Eigensolution of (M-2) and (M-3)
mc^2
4
2
0
-2
-4
$Q^{(2)}$
$R^{(2)}$
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
$r(\frac{mc}{\hbar})$

Fig. 6.11




Third Eigensolution of (M-2) and (M-3)
$\frac{mc^2}{g}$
2
0
-2
-4
$R^{(3)}$
$Q^{(3)}$
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
$r(\frac{mc}{\hbar})$

Fourth Eigensolution of (M-2) and (M-3)

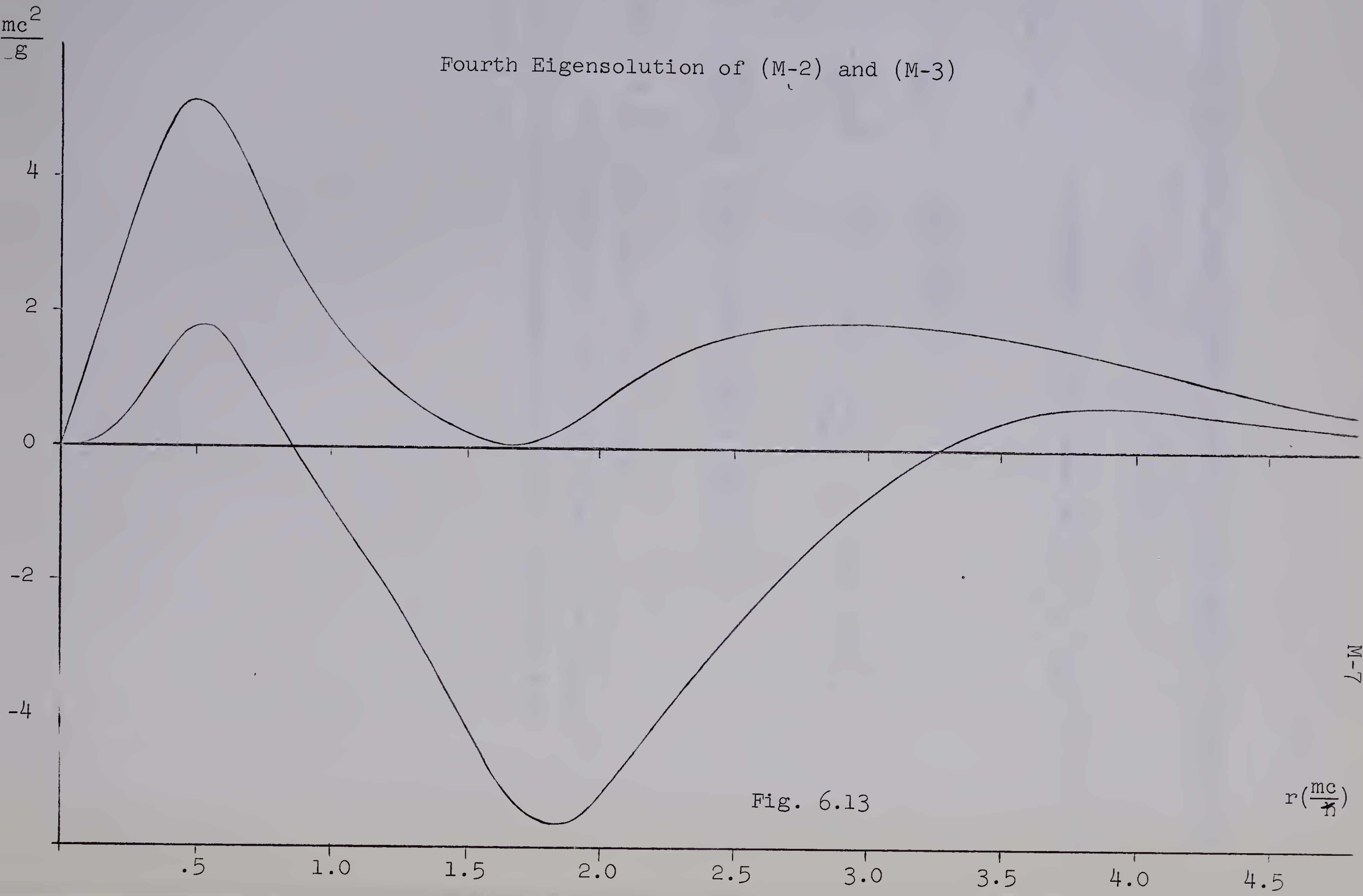


Fig. 6.13

APPENDIX N. VARIATIONAL APPROXIMATION TO ODD PARITY EIGENSOLUTION WITH $\phi_D = (r^2 + br^3)\ e^{-\alpha r} Y(\mu)$

All expressions are the same as for ϕ_B (Appendix L) except that

$$c_0 = \frac{4!}{(2\alpha)^5} \left(1 + 5B + \frac{15}{2}B^2\right) ,$$

$$c_2' = \frac{4!}{(2\alpha)^5} \frac{3}{2} \left(1 + 5B + 10B^2\right) + \frac{6!}{(2\alpha)^7} \left(1 + 7B + 14B^2\right) ,$$

and

$$c_4' = \frac{10!}{(4\alpha)^{11}} \left(1 + 11B + \frac{99}{2}B^2 + \frac{429}{4}B^3 + \frac{3003}{32}B^4\right) ,$$

where $B = \frac{b}{\alpha}$, $c_2' = c_0 c_2$ and $c_4' = c_0 c_4$.

The results for this trial function are given in table VI B, while the optimal form of $Y(\mu)$ is plotted in figure 6.14 .

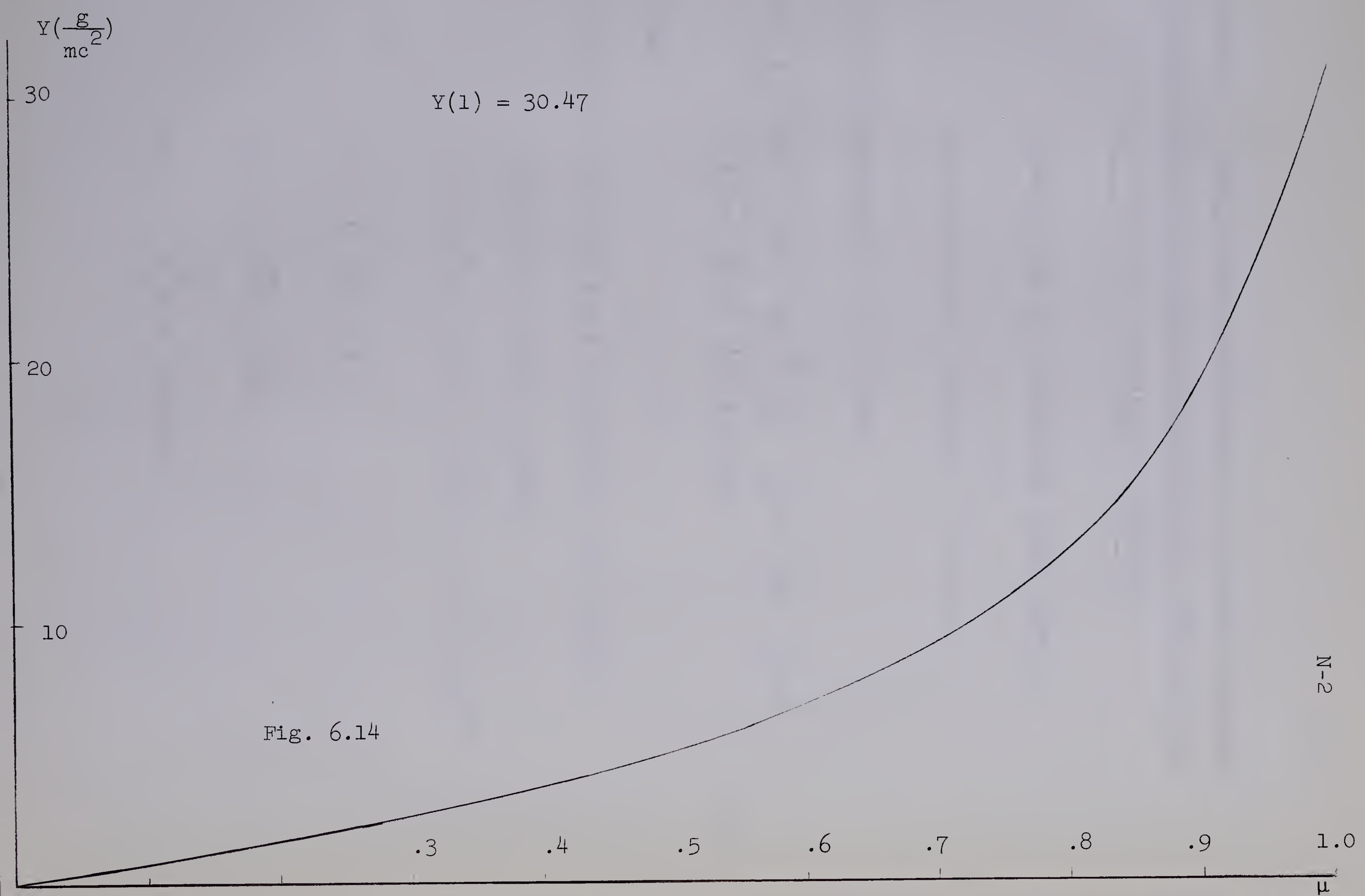


Fig. 6.14

N-2

APPENDIX O. VARIATIONAL APPROXIMATION TO POSSIBLE ODD PARITY NEUTRAL PARTICLELIKE EIGENSOLUTION USING THE TRIAL FUNCTION $\Phi_E = R(r)P_1(\mu) + Q(r)P_3(\mu) + S(r)P_5(\mu)$

Substituting the variational trial function

$$\Phi_E = R(r)P_1(\mu) + Q(r)P_3(\mu) + S(r)P_5(\mu)$$

into the Lagrangian (VI-2), we get

$$M = \frac{1}{2}\int_0^\infty r^2[\frac{1}{3}R'^2 + \frac{1}{7}Q'^2 + \frac{1}{11}S'^2 + \frac{1}{3}(1 + \frac{2}{r^2})R^2 + \frac{1}{7}(1 + \frac{12}{r^2})Q^2 + \frac{1}{11}(1 + \frac{30}{r^2})S^2 - \frac{1}{2}\, G(R,Q,S)]dr \qquad \text{(O-1)}$$

where

$$G(R,Q,S) = c_1R^4 + c_2Q^4 + c_3S^4 + 6(c_4R^2Q^2 + c_5R^2S^2 + c_6Q^2S^2) + 12(c_7R^2QS + c_8RQ^2S + c_9RQS^2) + 4(c_{10}R^3Q + c_{11}RQ^3 + c_{12}Q^3S + c_{13}RS^3 + c_{14}QS^3) ,$$

with

$$c_1 = \int_0^1 P_1^4 d\mu = \frac{1}{5} ,$$

$$c_2 = \int_0^1 P_3^4 d\mu = \frac{241}{5005} ,$$

$$c_3 = \int_0^1 P_5^4 d\mu = .0218450 ,$$

$$c_4 = \int_0^1 P_1^2 \ P_3^2 \ d\mu = \frac{23}{315} \ ,$$

$$c_5 = \int_0^1 P_1^2 \ P_5^2 \ d\mu = .0458430 \ ,$$

$$c_6 = \int_0^1 P_3^2 \ P_5^2 \ d\mu = .0254451 \ ,$$

$$c_7 = \int_0^1 P_1^2 \ P_3 \ P_5 \ d\mu = .0288600 \ ,$$

$$c_8 = \int_0^1 P_1 \ P_3^2 \ P_5 \ d\mu = .0299700 \ ,$$

$$c_9 = \int_0^1 P_1 \ P_3 \ P_5^2 \ d\mu = .0179820 \ ,$$

$$c_{10} = \int_0^1 P_1^3 \ P_3 \ d\mu = \frac{2}{35} \ ,$$

$$c_{11} = \int_0^1 P_1 \ P_3^3 \ d\mu = \frac{12}{385} \ ,$$

$$c_{12} = \int_0^1 P_3^3 \ P_5 \ d\mu = .0199800 \ ,$$

$$c_{13} = \int_0^1 P_1 \ P_5^3 \ d\mu = .0123406 \ ,$$

$$c_{14} = \int_0^1 P_3 \ P_5^3 \ d\mu = .0140417 \ .$$

$P_\ell(\mu)$ are the usual Legendre polynomials.

The Euler - Lagrange equations which must be satisfied by R, Q and S in order that (0-1) be extremised are:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \left(1+\frac{2}{r^2}\right)R + \frac{3}{4}G_R = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dQ}{dr}\right) - \left(1+\frac{12}{r^2}\right)Q + \frac{7}{4}G_Q = 0 \qquad (O\text{-}2)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dS}{dr}\right) - \left(1+\frac{30}{r^2}\right)S + \frac{11}{4}G_S = 0 ,$$

where $G_R = \frac{\partial G}{\partial R}$ and similarly for Q and S.

The resulting optimised trial functions ϕ_E satisfy identically the integral relations (V-4) and (V-6), for the same reason as the optimised trial functions ϕ_A and ϕ_C (see Appendix K). Thus, the expression (O-1) for the mass may be written as

$$M = \frac{1}{16\pi}\int \phi_E^4\, d^3x = \frac{1}{4}\int_0^\infty G(R,Q,S)r^2 dr.$$

The above set of coupled non-linear ordinary differential equations (O-2) can be solved numerically using the finite difference method described in Chapter V. Note that when r is large the solutions of (O-2) which are asymptotic to zero are of the form

$$R(r) \rightarrow Ak_1(r) = A\,\frac{e^{-r}}{r}\left(1+\frac{1}{r}\right) ,$$

$$Q(r) \rightarrow Bk_3(r) = B\,\frac{e^{-r}}{r}\left(1+\frac{6}{r}+\frac{15}{r^2}+\frac{15}{r^3}\right) ,$$

and

$$S(r) \rightarrow Ck_5(r) = C\,\frac{e^{-r}}{r}\left(1+\frac{15}{r}+\frac{105}{r^2}+\frac{420}{r^3}+\frac{945}{r^4}+\frac{945}{r^5}\right)$$

respectively, while for $r \ll 1$ R, Q and S are proportional to r, r^3 and r^5 respectively.

Numerical solution of the equations (0-2) indicates the existence of a discrete set of orthogonal (in the sense of (VI-1)) eigensolutions asymptotic to zero, corresponding to a discrete set of the constants A, B and C. The resulting set of optimised trial eigenfunctions $\phi_E^{(i)}$ then satisfy also the orthogonality relations (VI-1): For suppose (R_i, Q_i, S_i) and (R_j, Q_j, S_j) are two solutions of (0-2). Then multiplying the set of equations (0-2) for (R_i, Q_i, S_i) by R_j, Q_j and S_j respectively, performing the same calculation with i and j interchanged, subtracting the resulting integral relations and using Green's theorem we obtain

$$\int G_{R_i} R_j \, d^3x = \int G_{R_j} R_i \, d^3x$$

and similar results for Q and S. These relations then combine to yield

$$\int \phi_E^{(i)^3} \phi_E^{(j)} \, d^3x = \int \phi_E^{(i)} \phi_E^{(j)^3} \, d^3x \ ,$$

which is just the relation (VI-1) for the optimised trial eigenfunctions $\phi_E^{(i)}$.

The results for this trial function are given in tables VI-B and VIII-B, while the optimal forms of the (nodeless) functions R, Q and S are plotted in figure 6.15. The various integrals (p. 95) which are listed in table VIII-B, were evaluated numerically from:

$$E_x = \frac{1}{2}\int_0^\infty (r^2E_{x1} - 2rE_{x2} + E_{x3})\, dr$$

where $E_{x1} = \frac{2}{15}R'^2 + \frac{22}{315}Q'^2 + \frac{58}{1287}S'^2 - \frac{4}{35}R'Q' - \frac{40}{693}Q'S'$,

$$E_{x2} = \frac{2}{15}RR' - \frac{2}{35}RQ' + \frac{8}{35}QR' + \frac{4}{105}QQ' - \frac{20}{231}QS' + \frac{40}{345}SQ' + \frac{10}{429}SS' ,$$

$$E_{x3} = \frac{2}{3}R^2 + \frac{12}{7}Q^2 + \frac{30}{11}S^2 - E_{z3}$$

and $$E_z = \int_0^\infty (r^2E_{z1} + 2rE_{z2} + E_{z3})\, dr$$

where $E_{z1} = \frac{1}{5}R'^2 + \frac{23}{315}Q'^2 + \frac{59}{1287}S'^2 + \frac{4}{35}R'Q' + \frac{40}{693}Q'S'$,

$$E_{z2} = E_{x2} ,$$

$$E_{z3} = \frac{8}{15}R^2 + \frac{32}{35}Q^2 + \frac{400}{143}S^2 - \frac{16}{35}RQ - \frac{80}{77}QS$$

and $$E_r = \frac{2}{5}\int_0^\infty (r^5E_{r1} + 3r^4E_{r2} - r^4E_{r3})\, dr$$

where $$\begin{aligned} E_{r1} = {} & c_1RR'^3 + c_2QQ'^3 + c_3SS'^3 + 3[c_4(QR'^2Q' + RR'Q'^2) \\ & c_5(SR'^2S' + RR'S'^2) + c_6(SQ'^2S' + QS'^2Q')] + \\ & 3[c_7(R'^2(QS'+SQ') + 2RR'Q'S') + c_8(Q'^2(SR'+RS') \\ & + 2QR'Q'S') + c_9(S'^2(RQ'+QR') + 2SR'Q'S')] \\ & + c_{10}(3RQ'+QR')R'^2 + c_{11}(3QR'+RQ')Q'^2 + c_{12}(3QS' \\ & +SQ')Q'^2 + (c_{14}(3SQ'+QS') + c_{13}(3SR'+RS'))S'^2 , \end{aligned}$$

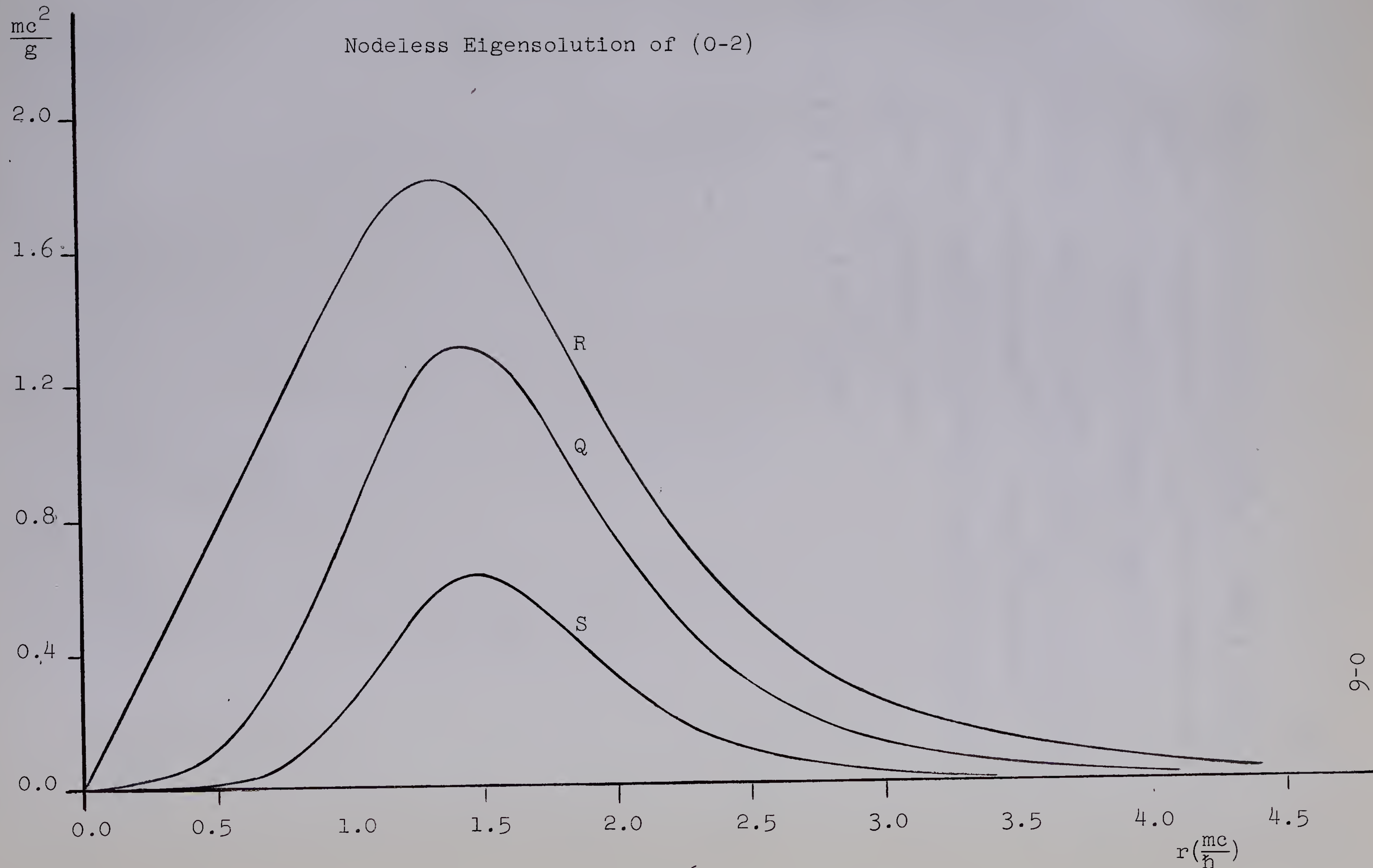

Nodeless Eigensolution of (0-2)
$\frac{mc^2}{g}$
2.0
1.6
1.2
0.8
0.4
0.0
R
Q
S
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
$r(\frac{mc}{\hbar})$

Fig. 6.15

$$E_{r2} = c_1 R^2 R'^2 + c_2 Q^2 Q'^2 + c_3 S^2 S'^2 + c_4(R^2 Q'^2 + 4RQR'Q' + Q^2 R'^2)$$
$$+ c_5(R^2 S'^2 + 4RSR'S' + S^2 R'^2) + c_6(Q^2 S'^2 + 4QSQ'S' + S^2 Q'^2)$$
$$+ c_7[4(RQR'S' + RSR'Q') + 2(R^2 Q'S' + QSR'^2)] +$$
$$c_8[4(RQQ'S' + QSR'Q') + 2(Q^2 R'S' + RSQ'^2)] + c_9[4(RSQ'S' +$$
$$QSR'S') + 2(S^2 R'Q' + RQS'^2)] + 2[c_{10}(R^2 R'Q' + RQR'^2) +$$
$$c_{11}(Q^2 R'Q' + RQQ'^2) + c_{13}(S^2 R'S' + RSS'^2) + c_{12}(Q^2 Q'S' +$$
$$QSQ'^2) + c_{14}(S^2 Q'S' + QSS'^2)]$$

and $E_{r3} = \frac{1}{3} R'^2 + \frac{1}{7} Q'^2 + \frac{1}{11} S'^2$.

APPENDIX P. SOLUTION OF FINITE DIFFERENCE EQUATION (V-13)

In the table below we list a result of iterative solution of the finite difference approximation (V-13) to (III-4). $\phi(r,\mu) = Ak_1(r)P_1(\mu)$ for $r \geq 5(\frac{\hbar}{mc})$, with $A = 10(\frac{\hbar c}{g})$. The mesh size used is $\delta r = .01(\frac{\hbar}{mc})$ and $\delta\mu = .05$.[1] $\phi(r,\mu)$ is plotted, as a function of μ, for some (arbitrarily chosen) fixed values of r in figures 6.15 and 6.16 .

r = 4

μ	ϕ
1.0	0.05722
0.9	0.05150
0.8	0.04578
0.7	0.04006
0.6	0.03434
0.5	0.02862
0.4	0.02289
0.3	0.01717
0.2	0.01145
0.1	0.00572
0.0	0.00000

r = 3

μ	ϕ
1.0	0.22045
0.9	0.19855
0.8	0.17660
0.7	0.15462
0.6	0.13260
0.5	0.11055
0.4	0.08847
0.3	0.06637
0.2	0.04425
0.1	0.02213
0.0	0.00000

---

1 Decreasing the mesh size (especially $\delta\mu$) had no significant effect on the results for $r \gtrsim 1$, but enhanced 'instability' for small values or r.

$r = 2$

μ	ϕ
1.0	0.95265
0.9	0.86825
0.8	0.78062
0.7	0.69001
0.6	0.59670
0.5	0.50103
0.4	0.40332
0.3	0.30397
0.2	0.20335
0.1	0.10189
0.0	0.00000

$r = 1$

μ	ϕ
1.0	2.16171
0.9	2.34495
0.8	2.51953
0.7	2.67184
0.6	2.77772
0.5	2.80386
0.4	2.68730
0.3	2.38340
0.2	1.82526
0.1	0.96306
0.0	0.00000

$r = .9$

μ	ϕ
1.0	1.86895
0.9	2.12883
0.8	2.40355
0.7	2.69074
0.6	2.96829
0.5	3.22666
0.4	3.32918
0.3	3.26777
0.2	2.78297
0.1	1.43204
0.0	0.00000

r = .8

μ	ϕ
1.0	1.37958
0.9	1.70520
0.8	2.05928
0.7	2.48208
0.6	2.90857
0.5	3.54832
0.4	3.85759
0.3	4.64753
0.2	5.05760
0.1	2.05416

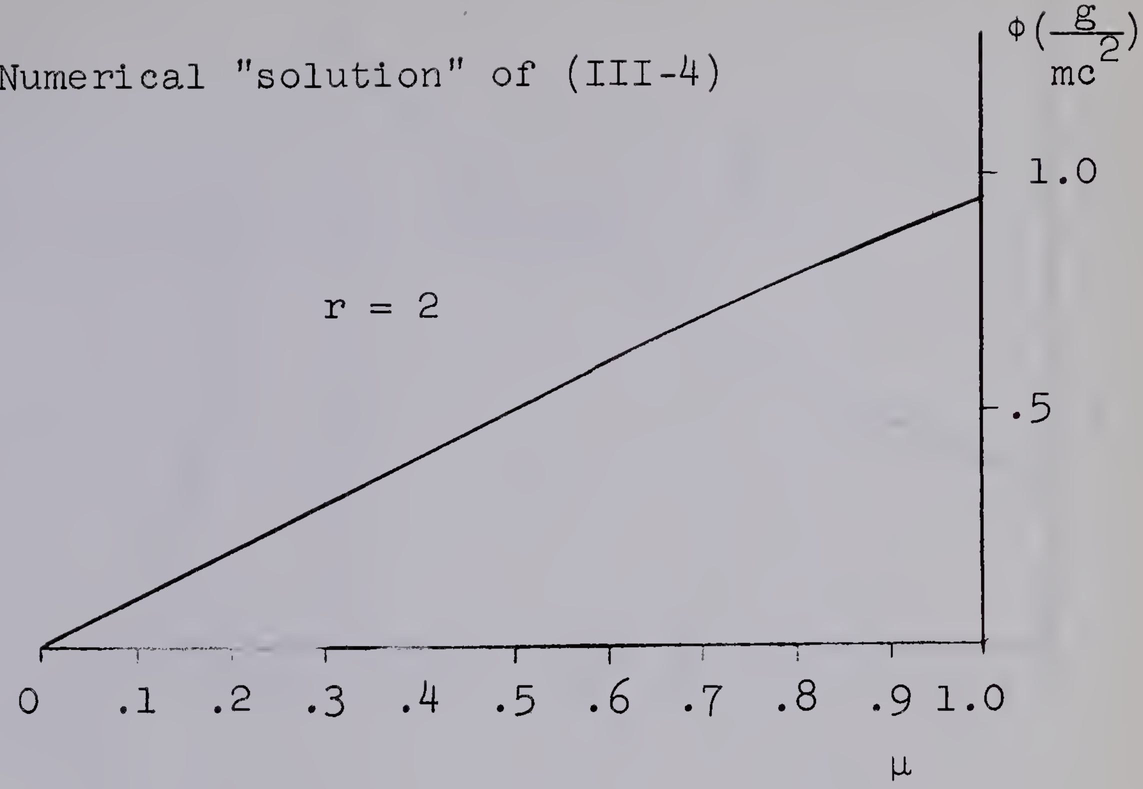
r = .7

μ	ϕ
1.0	0.61254
0.9	1.02327
0.8	1.34985
0.7	2.01833
0.6	2.14625
0.5	4.05149
0.4	2.37530
0.3	6.42154
0.2	12.55281
0.1	-0.21516
0.0	0.00000

r = .6

μ	ϕ
1.0	-0.80990
0.9	0.39154
0.8	-0.79439
0.7	2.98995
0.6	-3.84242
0.5	14.45970
0.4	-20.33843
0.3	9.71299
0.2	28.22908
0.1	-39.17139
0.0	0.00000

Numerical "solution" of (III-4)



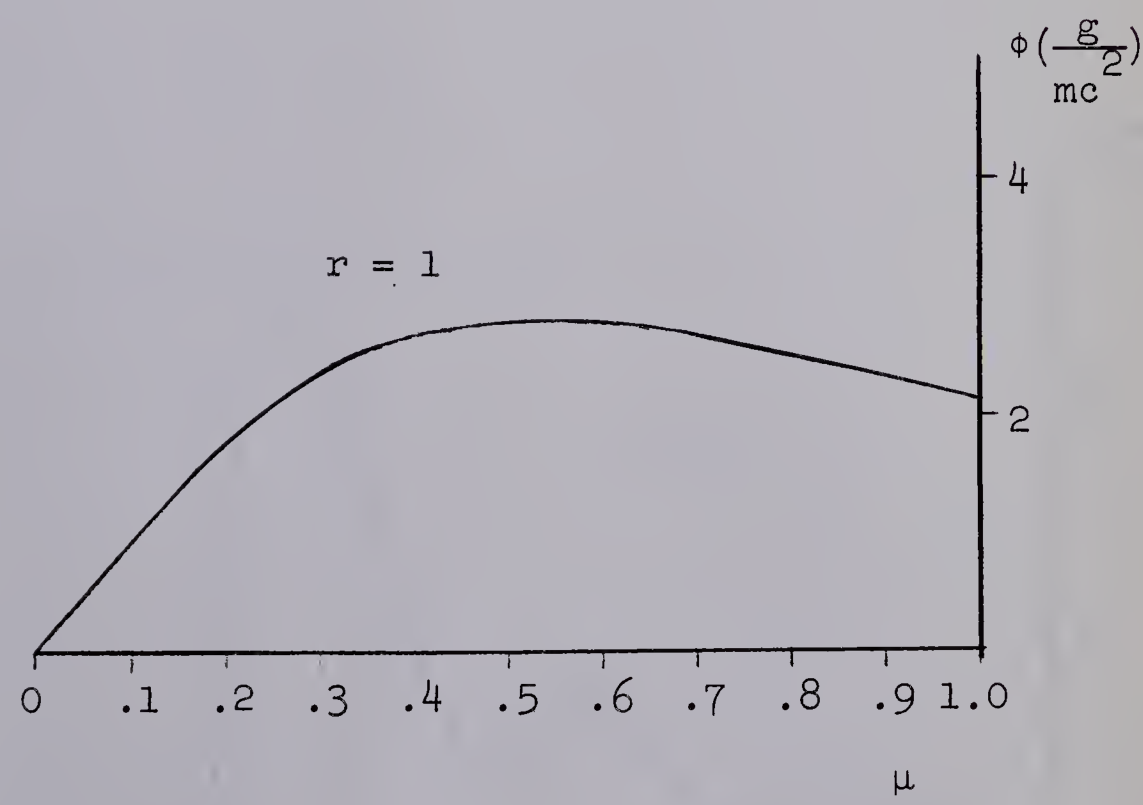
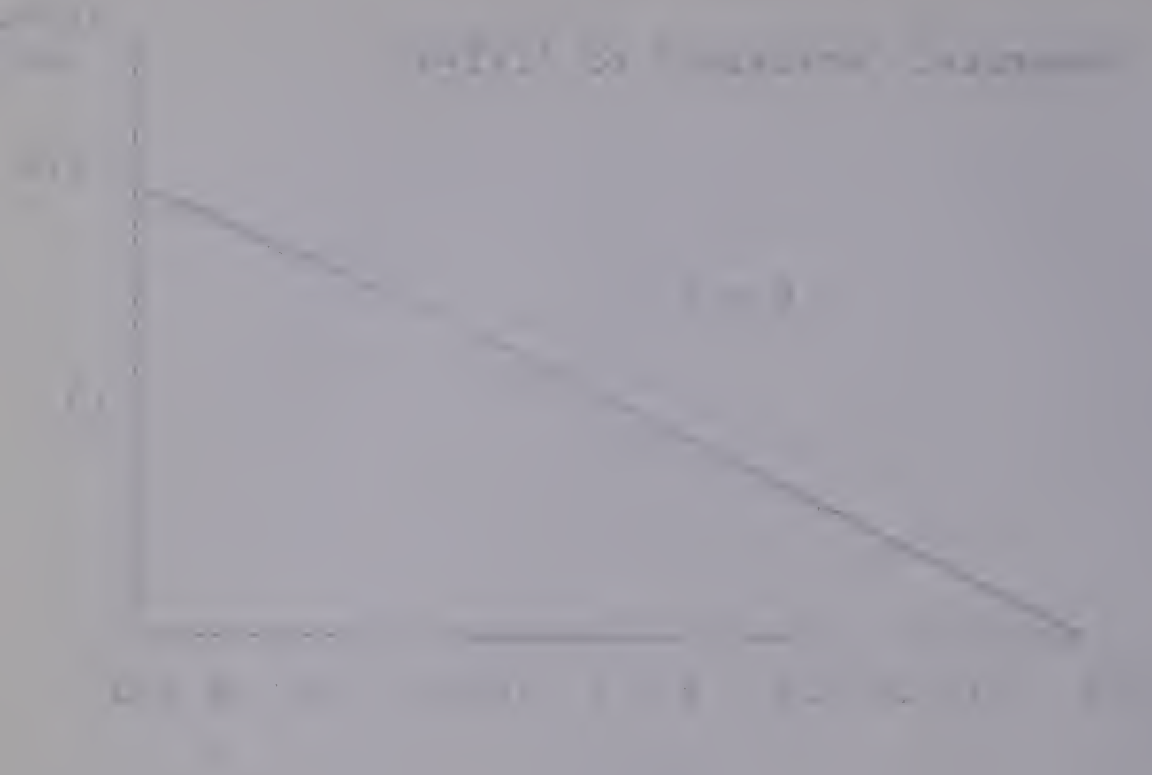
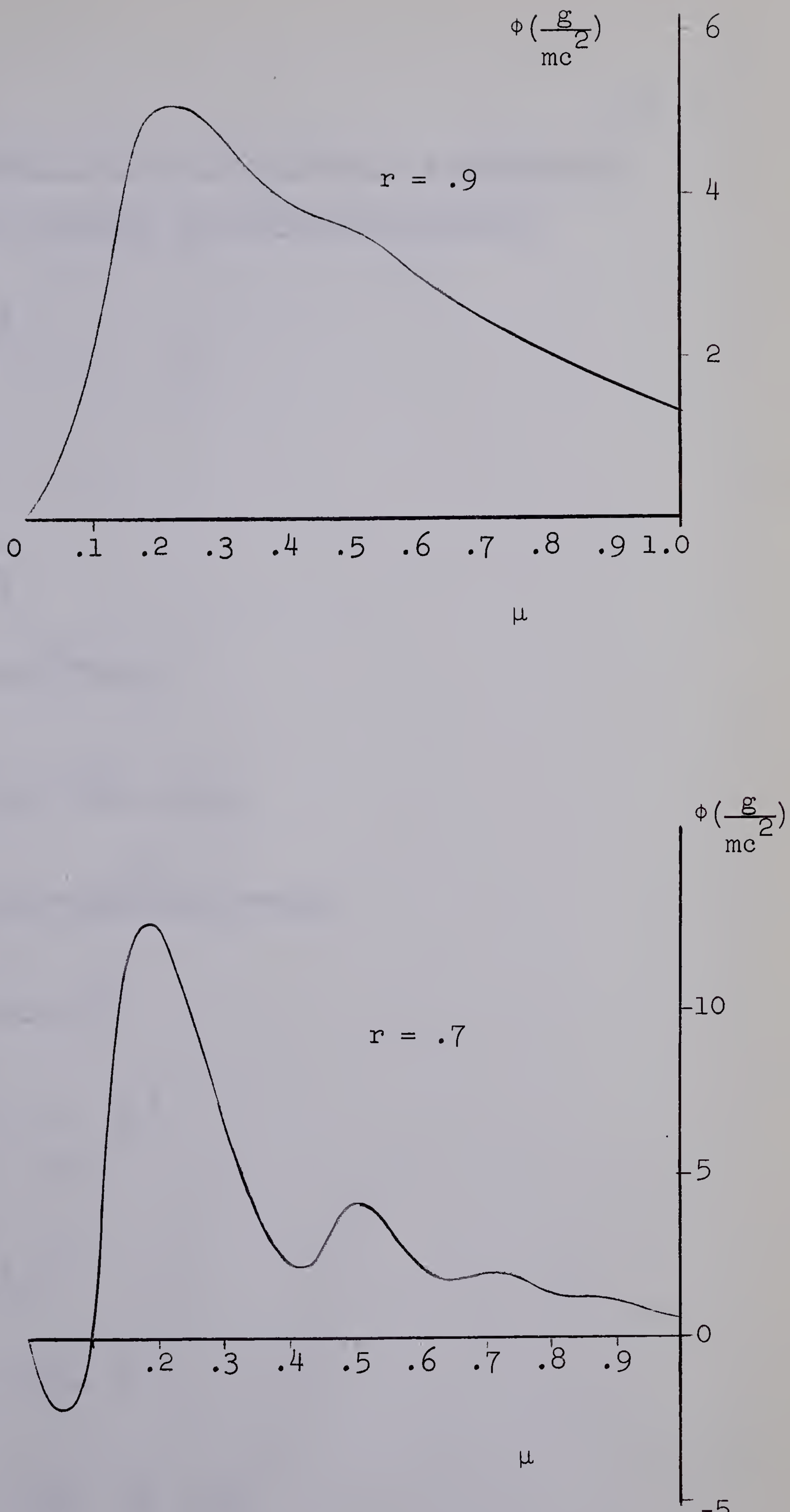


Fig. 6.16





$\phi(\frac{g}{mc^2})$
6
4
2
r = .9
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
μ
$\phi(\frac{g}{mc^2})$
10
5
0
-5
r = .7
.2
.3
.4
.5
.6
.7
.8
.9
μ

Fig. 6.17

APPENDIX Q. INTEGRALS OF TRIAL FUNCTIONS FOR SPHERICALLY SYMMETRIC CHARGED PARTICLELIKE SOLUTIONS

1. $f(x,\eta) = 1 - x^{\eta}$

$$c = 1$$

$$v = \eta$$

$$G = \frac{\eta^2}{2\eta+1}$$

$$G_2 = \frac{2\eta^2}{3(3+\eta)(3+2\eta)}$$

$$G_3 = \frac{2\eta^3}{(3+\eta)(3+2\eta)(3+3\eta)}$$

$$G_4 = \frac{8\eta^4}{(3+\eta)(3+2\eta)(3+3\eta)(3+4\eta)}$$

2. $f(x,\eta) = 1 - 2x + \eta x^2$

$$\eta c = 1 - (1 - \eta)^{\frac{1}{2}}$$

$$v = 2(1 - \eta c)$$

Defining $u = \eta c$

and $H(0) = \frac{1}{3} c^3$

$$H(1) = c^4(\tfrac{1}{5}u - \tfrac{1}{2})$$

$$H(2) = c^5(\tfrac{1}{7}u^2 - \tfrac{2}{3}u + \tfrac{4}{5})$$

$$H(3) = c^6(\tfrac{1}{9}u^3 - \tfrac{3}{4}u^2 + \tfrac{12}{7}u - \tfrac{4}{3})$$

$$H(4) = c^7(\tfrac{1}{11}u^4 - \tfrac{4}{5}u^3 + \tfrac{8}{3}u^2 - 4u + \tfrac{16}{7}) ,$$

then

$$G = 4c^3(\tfrac{1}{5}u^2 - \tfrac{1}{2}u + \tfrac{1}{3})$$

and

$$G_k = \sum_{n=0}^{k} \binom{k}{n} H(n) ,$$

$\binom{k}{n}$ being the usual binomial coeficients.

<u>3. $f(x,\eta,\sigma) = 1 - 2x^{\sigma} + \eta x^{2\sigma}$</u>

Setting $z = c^{\sigma}$,

$$\eta z = 1 - (1 - \eta)^{\frac{1}{2}} ,$$

$$v = \frac{2\sigma z}{c}(1 - \eta z) ,$$

and defining

$$g_n = \frac{z^n c^3}{n\sigma + 3} ,$$

$$G_k = \sum_{n=0}^{2k} \left[{k \atop n}\right] g_n ,$$

where the coeficients $\left[{k \atop n}\right]$ are defined by

$$(1 - 2u + \eta u^2)^k = \sum_{n=0}^{2k} \left[{k \atop n}\right] u^n .$$

4. $f(x,\eta) = \frac{1}{x^2+1} - \eta$

$$c = \left(\frac{1}{\eta} - 1\right)^{\frac{1}{2}}$$

$$v = 2c\eta^2$$

Defining $H(1) = \tan^{-1}c$

and $$H(n) = \frac{1}{2(n-1)}[c\eta^{n-1} + (2n - 3)\, H(n-1)] \text{ when } n>1,$$

$$F(0) = \frac{1}{3}\, c^3$$

and $$F(n) = \frac{1}{2n-3}[H(n) - c\eta^{n-1}] \text{ when } n>0,$$

$$G = 4[F(4) - \frac{1}{3}\, c^3\eta^3]$$

while $$G_k = \sum_{n=0}^{k} (-1)^n \binom{k}{n} F(k-n)\eta^n,$$ where $\binom{k}{n}$ are binomial coeficients.

5. $f(x,\eta,\sigma) = \cos(x+\eta) + \sigma$

$$s = c + \eta = \frac{\pi}{2} + \sin^{-1}\sigma$$
$$v = (1 - \sigma^2)^{\frac{1}{2}}$$

$$G = \frac{1}{2}sv(v+s\sigma) + \frac{1}{6}s^3 - \frac{1}{2}\eta \sin\eta(\sin\eta-\eta\cos\eta) - \frac{1}{6}\eta^3 + \frac{1}{2}(\eta^2 - \frac{1}{2})(c + v\sigma$$

$$+ \sin\eta \cos\eta) - \frac{1}{2}\eta[v(v+2s\sigma) + s^2 - \sin\eta(\sin\eta-2\cos\eta) - \eta^2] \, .$$

Defining

$$F(m+1,0) = \frac{1}{m+2}(s^{m+2} - \eta^{m+2}),$$

$$F(m+1,1) = s^{m}(sv-(m+1)\sigma) - \eta^{m}((m+1)\cos\eta + \eta\sin\eta) - (m+1)mF(m-1,1)$$

for $m = -1,0,1,2\ldots$

and

$$F(1,n+1) = \frac{1}{(n+1)^2}\{(-\sigma)^n[(n+1)sv-\sigma] - \cos^n\eta[\cos\eta + (n+1)\eta\sin\eta]\} + \frac{n}{n+1}F(1,n-1),$$

$$F(2,n+1) = \frac{1}{(n+1)^2}\{s(-\sigma)^n[(n+1)sv-2\sigma] - \eta\cos^n\eta[2\cos\eta+(n+1)\eta\sin\eta]\} + \frac{n}{n+1}F(2,n-1) - \frac{2}{(n+1)^2}F(0,n+1) \quad \text{when } n = 2,3\ \ldots$$

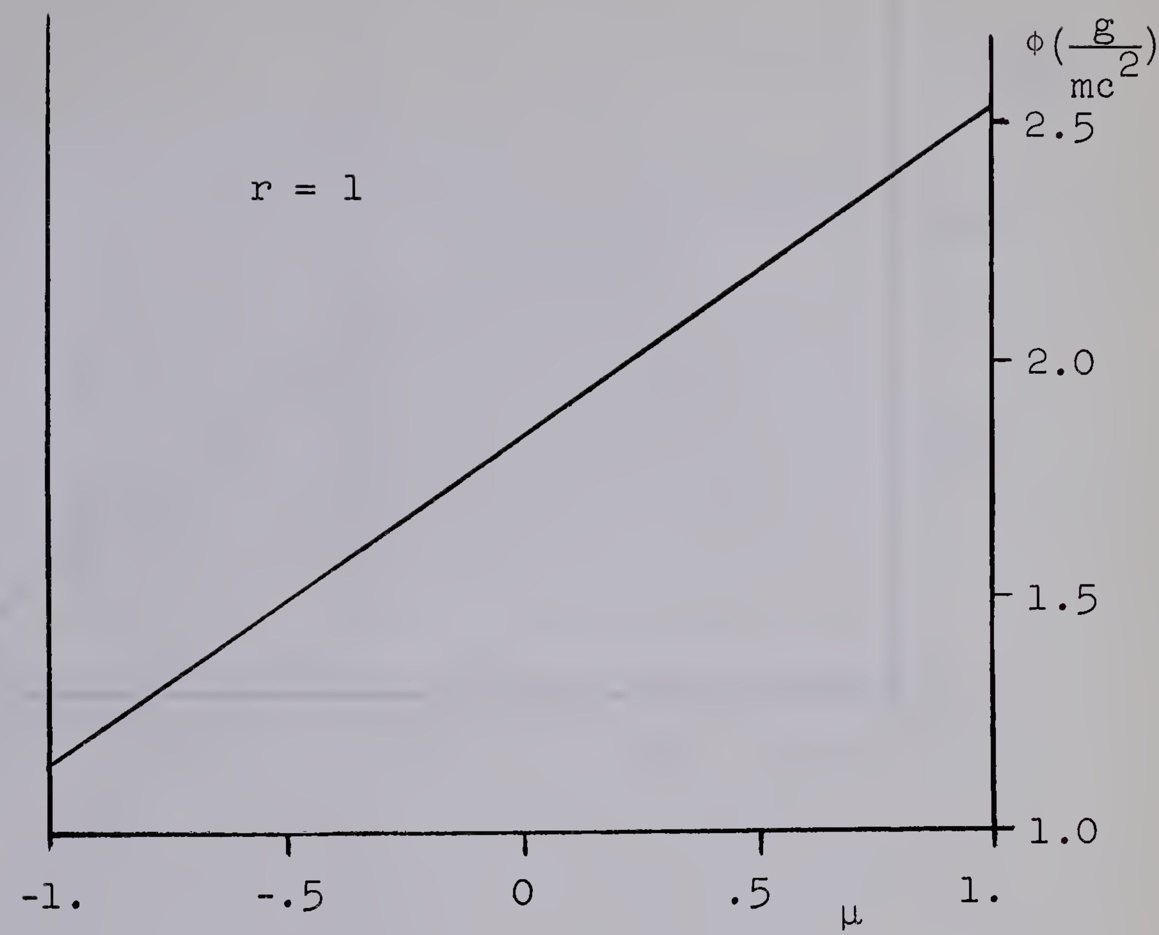
Also, if $H(n) = F(2,n) - 2\eta F(1,n) + \eta^2 F(0,n)$ then

$$G_k = \sum_{n=0}^{k} \binom{k}{n} H(k-n)\,\sigma^n\ ,$$ $\binom{k}{n}$ being binomial coeficients.

APPENDIX R. APPROXIMATE 'SOLUTION' OF (VII-13)

Figures 7.7 and 7.8 below are plots of cross-sections (for certain, arbitrarily chosen, fixed values of r) of a numerically obtained "solution" of (VII-13). The method used in obtaining these results is the method of iteratively solving the finite difference approximation to (VII-13); this method is described in Chapter V.

For the example plotted below $a = -2.177 \frac{mc^2}{g}$, $b = 2.77 \frac{\hbar c}{g}$ and $p = .1043 \frac{\hbar^2}{gm}$. The mesh size used is $\delta r = .01 \frac{\hbar}{mc}$ and $\delta\mu = .05$. The results were found to be essentially unaltered when the mesh size was varied (decreased) except for the region where $r \lesssim .4$, where the unstable "oscillations" which are apparent (fig. 7.8) grew as the mesh size was decreased (especially $\delta\mu$).


r = 1
$\phi(\frac{g}{mc^2})$
2.5
2.0
1.5
1.0
-1.
-.5
0
.5
μ
1.

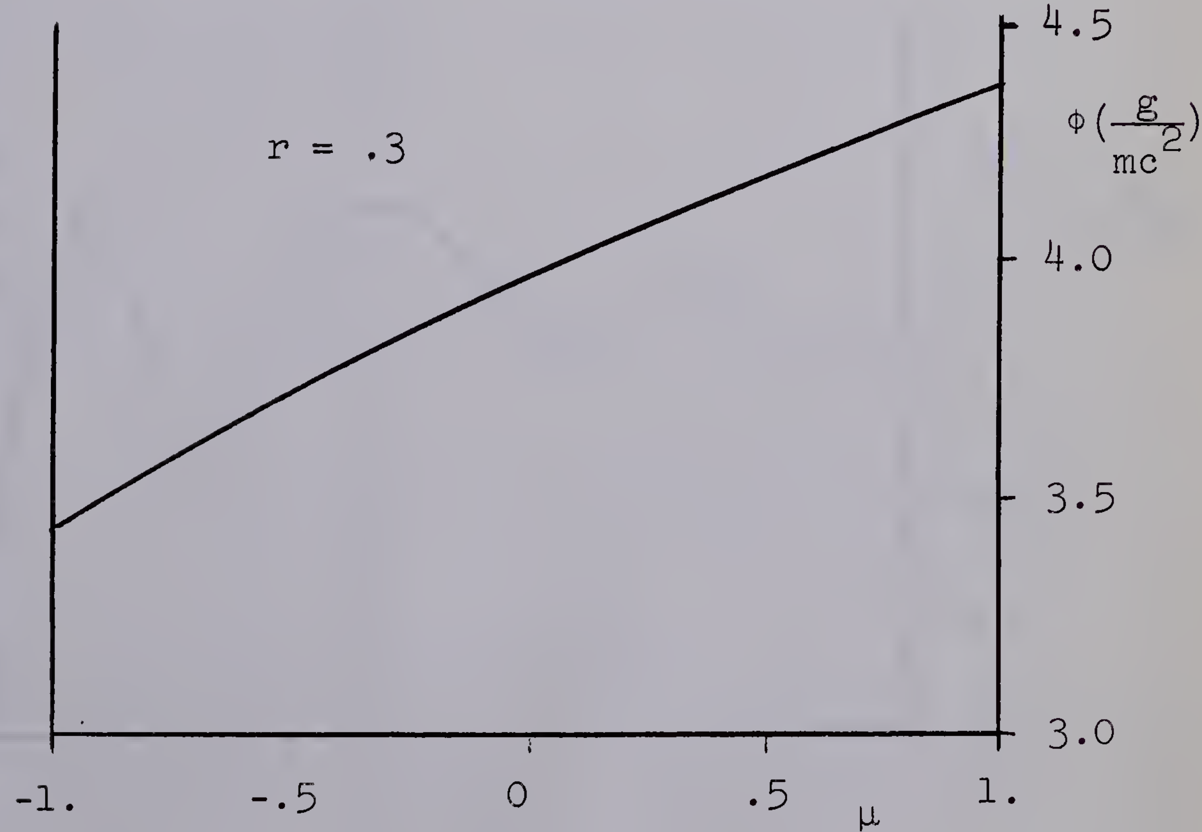
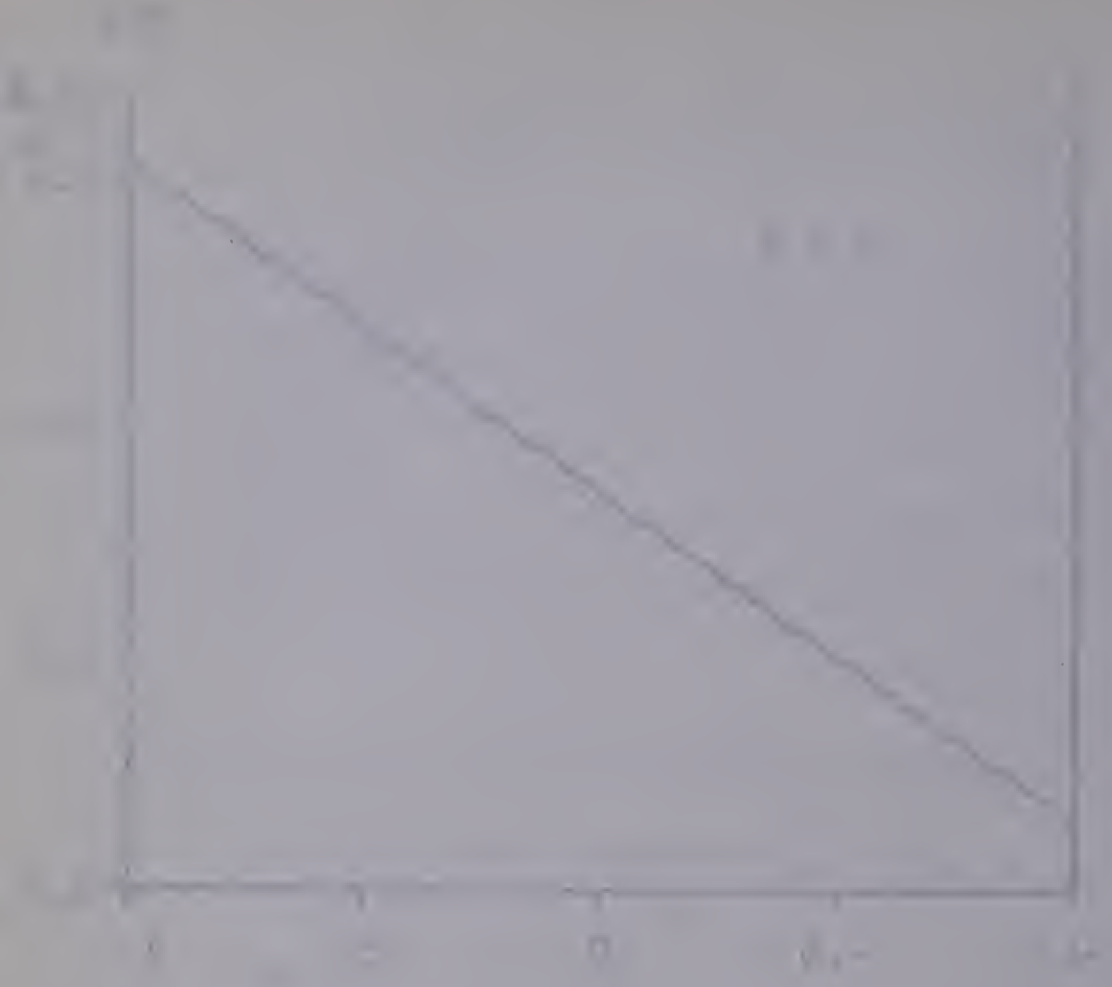

r = .3
4.5
$\phi(\frac{g}{mc^2})$
4.0
3.5
3.0
-1.
-.5
0
.5
μ
1.

Fig. 7.7



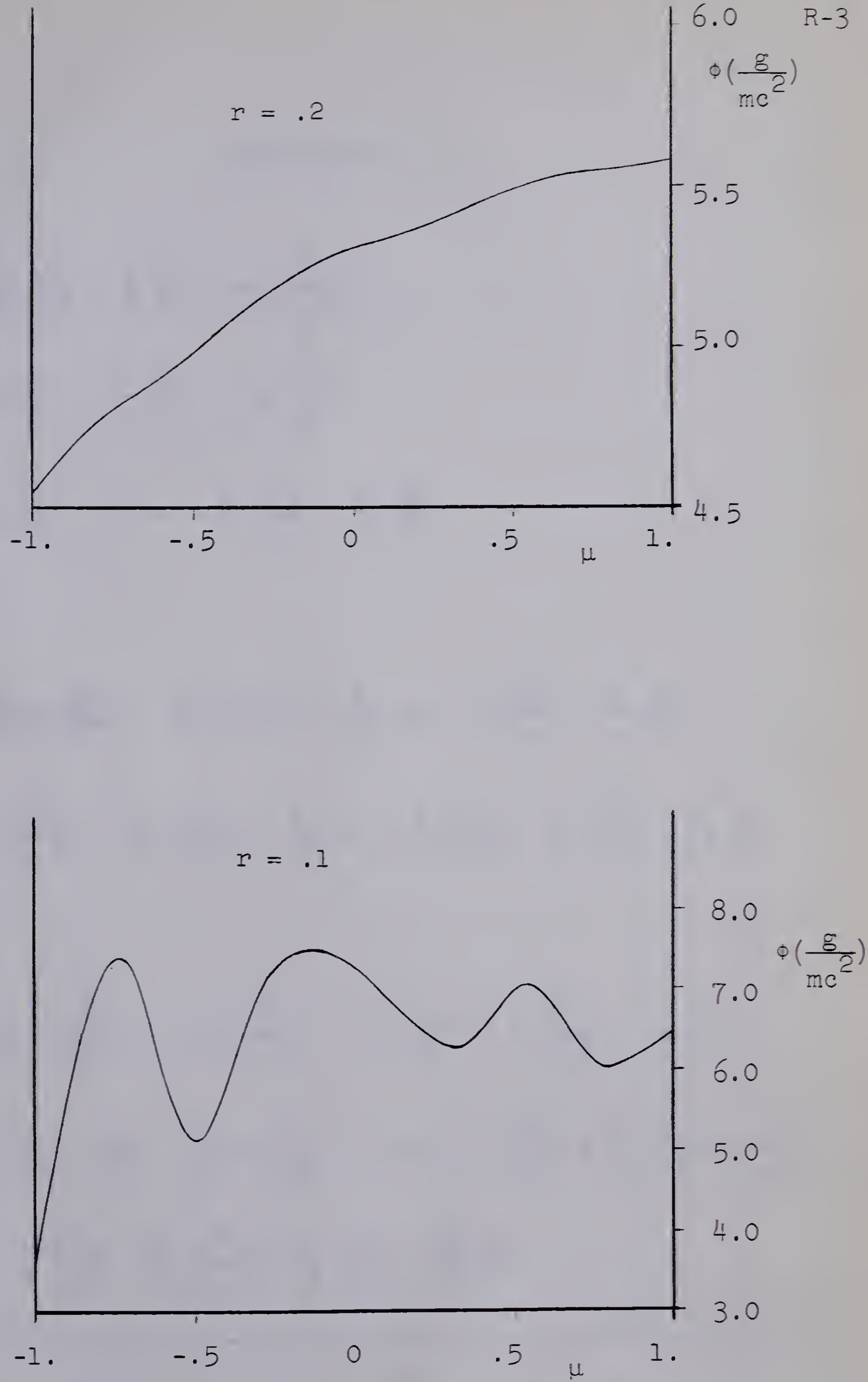


Fig. 7.8

APPENDIX S

$$\frac{\partial}{\partial r}(\Delta u) = \frac{\partial}{\partial r}\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \Omega^2}\right]^1$$

$$= \Delta\left(\frac{\partial u}{\partial r}\right) - \frac{2}{r^2}\frac{\partial u}{\partial r} - \frac{2}{r^3}\frac{\partial^2 u}{\partial \Omega^2}$$

$$= \Delta\left(\frac{\partial u}{\partial r}\right) - \frac{2}{r}\Delta u + \frac{2}{r}\frac{\partial^2 u}{\partial r^2} + \frac{2}{r^2}\frac{\partial u}{\partial r} .$$

Therefore

$$\frac{\partial^2(\Delta u)}{\partial r^2} = \frac{\partial}{\partial r}\left(\Delta\left(\frac{\partial u}{\partial r}\right)\right) - \frac{2}{r}\frac{\partial}{\partial r}(\Delta u) + \frac{2}{r^2}\Delta u + \frac{2}{r}\frac{\partial^3 u}{\partial r^3} - \frac{4}{r^3}\frac{\partial u}{\partial r}$$

$$= \Delta\left(\frac{\partial^2 u}{\partial r^2}\right) - \frac{4}{r}\Delta\left(\frac{\partial u}{\partial r}\right) + \frac{6}{r^2}\Delta u + \frac{4}{r}\frac{\partial^3 u}{\partial r^3} - \frac{2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{8}{r^3}\frac{\partial u}{\partial r} ,$$

and

$$\frac{\partial^3(\Delta u)}{\partial r^3} = \frac{\partial}{\partial r}\Delta\left(\frac{\partial^2 u}{\partial r^2}\right) - \frac{4}{r}\left[\Delta\left(\frac{\partial^2 u}{\partial r^2}\right) - \frac{2}{r}\Delta\left(\frac{\partial u}{\partial r}\right) + \frac{2}{r}\frac{\partial^3 u}{\partial r^3} + \frac{2}{r^2}\frac{\partial^2 u}{\partial r^2}\right]$$

$$+ \frac{4}{r^2}\Delta\left(\frac{\partial u}{\partial r}\right) + \frac{6}{r^2}\left[\Delta\left(\frac{\partial u}{\partial r}\right) - \frac{2}{r}\Delta u + \frac{2}{r}\frac{\partial^2 u}{\partial r^2} + \frac{2}{r^2}\frac{\partial u}{\partial r}\right] - \frac{12}{r^3}\Delta u$$

$$+ \frac{4}{r}\frac{\partial^4 u}{\partial r^4} - \frac{6}{r^2}\frac{\partial^3 u}{\partial r^3} - \frac{4}{r^3}\frac{\partial^2 u}{\partial r^2} + \frac{24}{r^4}\frac{\partial u}{\partial r} ,$$

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1 $\frac{\partial^2 u}{\partial \Omega^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}$.

That is

$$\frac{\partial^3(\Delta u)}{\partial r^3} = \Delta\left(\frac{\partial^3 u}{\partial r^3}\right) - \frac{6}{r}\,\Delta\left(\frac{\partial^2 u}{\partial r^2}\right) + \frac{18}{r^2}\,\Delta\left(\frac{\partial u}{\partial r}\right) - \frac{24}{r^3}\,\Delta u + \frac{6}{r}\,\frac{\partial^4 u}{\partial r^4}$$

$$- \frac{12}{r^2}\,\frac{\partial^3 u}{\partial r^3} + \frac{36}{r^4}\,\frac{\partial u}{\partial r} \,.$$